This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

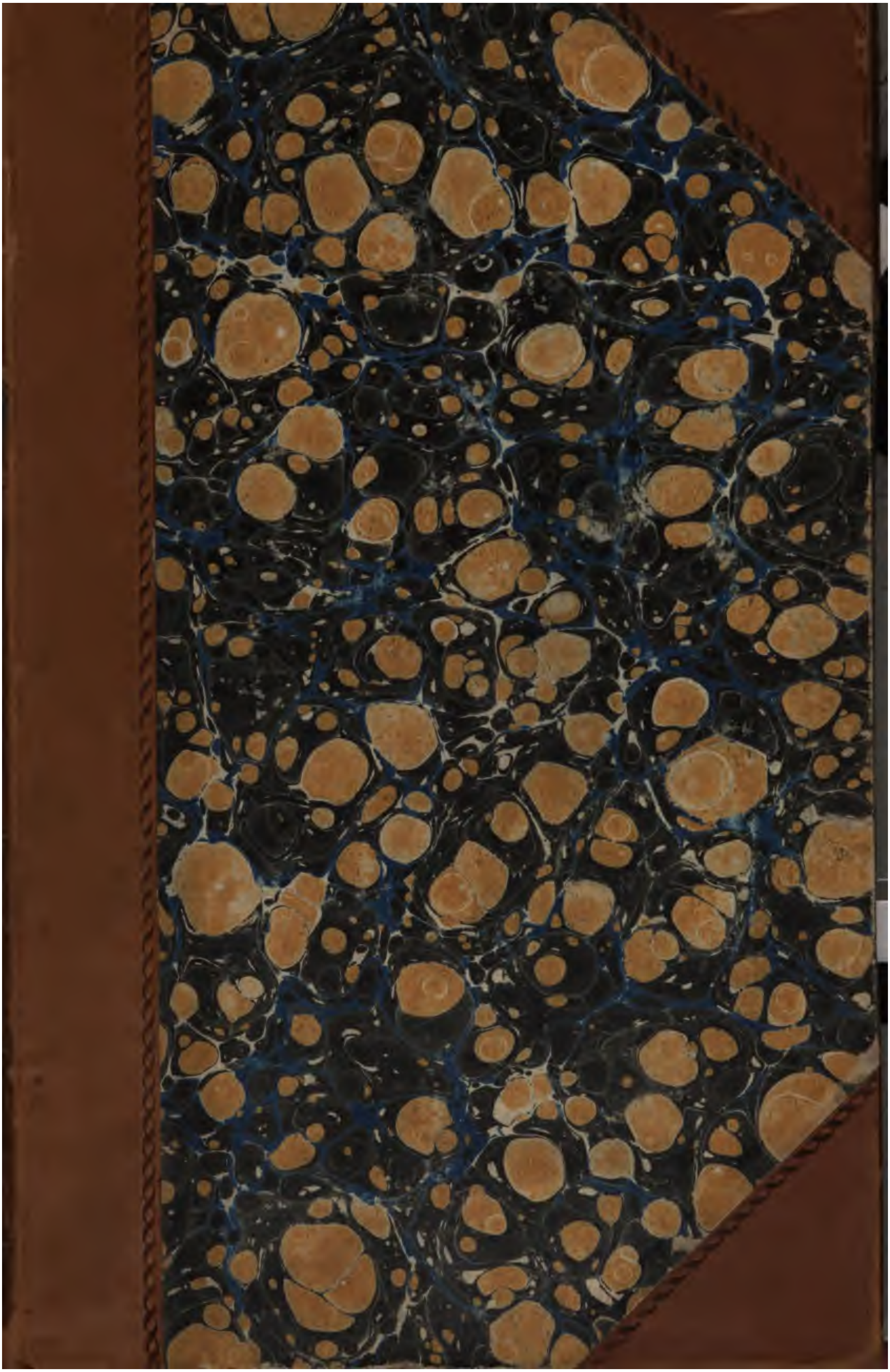
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`



600006294R

600006294R

THE

ELEMENTS

OF THE

THEORY OF MECHANICS.

BIBLIOTHECA BODL.

BY THE

REV. ROBERT WALKER, M. A.

OF WADHAM COLLEGE.

OXFORD
D A TALBOYS
1830

898.

PREFACE.

It is the object of the present treatise to exhibit, in a rigorous and connected form, the principles of Mechanical Science. The works in this department already published in this country are, it is conceived, either limited by the method of demonstration, or deficient in pointing out the connexion of the various parts of the subject. In the present treatise, therefore, the powerful aids of analysis have been introduced; and with the view of marking the distinction between that which is important as theory, and that which is the application of the theory, each of the great divisions of the subject is subdivided into two parts; the first part containing that which rests upon the assumed methods of estimating force, and which will consequently stand upon that, as an hypothesis, were matter differently constituted; the second containing the application of these principles to matter as it is. The application has been carried only so far as seemed necessary to exhibit the importance and object of the results of the theory. Examples have been purposely avoided as tending to draw off the attention from the dependence of one principle upon another, and therefore excluded by the design of the treatise: but, should this attempt be favourably received, the author proposes to prepare another volume, which shall be, as it were, a companion to the present. This second volume shall contain deductions from the principles here established, a selection of examples with their solutions, either complete, or partly so, or merely with their results, as may appear necessary, and any historical illustration or remark which may throw light or interest upon the subject.

In a work like the present there can be little room for originality, and originality has not been aimed at. Every work, which was thought likely to assist, has been consulted; and whatever appeared to fall in with the design, has been borrowed. The names of Pontécoulant, Poisson, Boucharlat, Venturoli, Lloyd, and Whewell must be mentioned as authors, whose works were constantly at hand; and to the first of these the present treatise is most largely indebted.

For the convenience of students, asterisks are prefixed in the Table of Contents to those articles and chapters which may be omitted on a first reading.

WADHAM COLLEGE,
July 1830.

CONTENTS.

INTRODUCTION.

1. Mechanics. Subject divided into Statics and Dynamics.
2. *Motion* and *Rest.*
3. Meaning attached to *force.*
4. Force measured by its *effect,* which depends on the *intensity,* the *point of application,* and the *direction.*
5. *Body,* an assemblage of points.

STATICS.

SECTION I.

CHAP. I.

FORCES APPLIED TO A POINT.

6. Two equal and opposite forces are in equilibrio.
7. Any two forces inclined at an angle are not in equilibrio.
8. Meaning of the words *resultant* and *component.*
9. Parallelogram of forces.
10. Two forces and their resultant represented by the sides of a triangle.
11. Any number of forces represented by a series of parallelograms or a polygon.

12. Parallelepiped of forces.

13. Analytical formulæ for the resultant of any number of forces acting on a point;

$$X = \Sigma . p \cos . \alpha, \quad Y = \Sigma . p \cos . \beta, \quad Z = \Sigma . p \cos . \gamma;$$

$$R = \sqrt{(X^2 + Y^2 + Z^2)};$$

$$\cos . A = \frac{X}{R}, \quad \cos . B = \frac{Y}{R}, \quad \cos . \Gamma = \frac{Z}{R}.$$

14. Modification of formulæ, when the forces are in the same plane.

15. *Moment* of a force with reference to a *point*. Moment of resultant is equal to the sum of the moments of the components.

16. Modification of formulæ, when the forces are in the same line.

17. Convenient formula for two forces,

$$R^2 = p'^2 + p''^2 + 2p'p'' \cos . \theta.$$

18. Conditions of equilibrium,

$$X = 0, \quad Y = 0, \quad Z = 0.$$

19. Formulæ for the equilibrium of a point on a curve,

$$N = \sqrt{(X^2 + Y^2 + Z^2)},$$

$$X dx + Y dy + Z dz = 0.$$

20. Modification of formulæ when the curve is in the vertical plane.

CHAPTER II.

PARALLEL FORCES.

21. The resultant of two parallel forces is equal to their sum or difference, is in the same direction, and its point of application is determined by the proportion

$$p' : p'' :: q'' : q'.$$

Case of two equal and contrary, but not opposite forces.

22. Resultant of any number of parallel forces.

23. *Centre* of parallel forces.

24. Formulæ for finding the position of the centre of parallel forces,

$$x_{,} = \frac{\Sigma.px}{\Sigma.p}, \qquad y_{,} = \frac{\Sigma.py}{\Sigma.p}, \qquad z_{,} = \frac{\Sigma.pz}{\Sigma.p}.$$

25. Modification of formulæ, when the forces are in the same plane.

26. Modification of formulæ, when the forces are in the same line.

27. *Moment* of a force with reference to a *plane*.

28. Moment of resultant equal to the sum of the moments of the components.

29. Conditions of equilibrium,

1st. Sum of the forces equal to zero,

2ndly. Sum of their moments with reference to two parallel planes, each equal to zero.

30. One force equal and opposite to the others, in case of equilibrium.

31. If equilibrium subsist in one, it does in all directions.

*CHAPTER III.

FORCES ACTING IN SPACE.

32. Formulæ for the effects of forces any how situated,

$$X = \Sigma . p \cos . \alpha, \qquad Y = \Sigma . p \cos . \beta, \qquad Z = \Sigma . p \cos . \gamma,$$

$$L = \Sigma . p (\cos . \alpha . y - \cos . \beta . x),$$

$$M = \Sigma . p (\cos . \gamma . x - \cos . \alpha . z),$$

$$N = \Sigma . p (\cos . \beta . z - \cos . \gamma . y).$$

33. Conditions of equilibrium,

$$X = 0, \quad Y = 0, \quad Z = 0; \quad L = 0, \quad M = 0, \quad N = 0.$$

34. Modification of formulæ, when the forces meet in a point.
35. Modification, when the forces are parallel.
36. Modification, when the forces are in the same plane.
37. Formula of condition for a single resultant,

$$LZ + MY + NX = 0.$$

38. Magnitude of resultant, as in 13.
39. Case where $X = 0$, $Y = 0$, and $Z = 0$.
40. Case where $X = 0$, $Y = 0$, and Z does not $= 0$.
41. Case where only $X = 0$.
42. Case where $L = 0$, $M = 0$, $N = 0$, but either X, Y, or Z not $= 0$.
43. Case where one or more of the moments L, M, N, and one or more of the forces X, Y, Z do not $= 0$.
44. If the equation is not satisfied, no single resultant.

CHAPTER IV.

SYSTEMS WITH FIXED POINTS.

45. Force applied to a fixed point.
46. Force applied to a point constrained to a move in a line.
47. Force applied to a point on a plane.
48. Direction of the *reaction* in each case.
* 49. Conditions of equilibrium when there is one fixed point,

$$L = 0, \quad M = 0, \quad N = 0.$$

* 50. Conditions of equilibrium when there are two fixed points; the line joining the two points is an axis, suppose the axis of Z, and then $L = 0$.
* 51. If motion may take place along the axis, $Z = 0$ is another condition.
* 52. If three fixed points not in the same line, no motion can ensue.
* 53. Conditions of equilibrium, when the point is on a plane.

$$X = 0, \quad Y = 0; \quad L = 0, \quad M = 0, \quad N = 0,$$

assuming the point as origin, and plane as the plane of x, y.
* 54. If two points are on a plane, two parallel forces of reaction.
* 55. Only the two extreme points in the same line necessary.
* 56. If three points not in the same line, three parallel forces of reaction.
* 57. Pressures on the points.

CHAPTER V.

SYSTEMS OF VARIABLE FORM.

58. Conditions for a *rigid system*, not sufficient for a variable one.

SECTION II.

CHAP. I.

CENTRE OF GRAVITY.

59. Force of *gravity*.
60. Gravity may be considered constant for the same place.
61. *Weight* of a body.
62. *Density* of a body.
63. $W = g\,\mathrm{D}\,\mathrm{V}$ generally.
64. *Centre* of gravity.
65. Formulæ for finding the centre of gravity.
66. By the integral calculus, the formulæ become

$$x_{,} = \frac{\int x\,dv}{\int dv}, \quad y_{,} = \frac{\int y\,dv}{\int dv}, \quad z_{,} = \frac{\int z\,dv}{\int dv},$$

67. Formulæ for a plane curve,

$$x_{,} = \frac{\int x\,ds}{\int ds}, \quad y_{,} = \frac{\int y\,ds}{\int ds}.$$

If curve is symmetrical, one equation is sufficient.

68. For an area,

$$x_{,} = \frac{\int x y\,dx}{\int y\,dx}, \quad y_{,} = \frac{\int y^2\,dx}{\int y\,dx}.$$

If area is symmetrical, one is sufficient.

69. For a surface of revolution round axis of x,

$$x_{,} = \frac{\int x y\, d s}{\int y\, d s}.$$

70. For a solid of revolution,

$$x_{,} = \frac{\int y^2 x\, d x}{\int y^2 d x}.$$

71. The centre of gravity of an uniform straight line.
72. The centre of gravity of a parallelogram.
73. Similarly, of a triangle.
74. If the centre of gravity is supported, the body is at rest in all positions.
75. The centre of gravity must be vertically above or below the point of suspension.
76. Equilibrium, *stable, neutral,* and *unstable.*
77. If the centre of gravity is below the point of suspension, the equilibrium is stable; if at that point, neutral; if above, unstable.
78. Practical method of finding the centre of gravity.
79. Pressure on a fixed point equal to the weight of the body.
80. If there are two fixed points, the centre of gravity must be in the vertical plane through them.
81. If three fixed points, the body is fixed.
82. If the body is on a horizontal plane, and touch in one point, the reaction must sustain the weight, and we have stable, neutral, and unstable equilibrium.
83. If it touch the plane in two points, the weight must be in the same vertical plane as the reactions.
84. If more than two points in the same line, only the extreme ones necessary.

85. If it touch the plane in three points, the direction of weight must fall within the triangle joining them.
86. If it touch in more than three, the problem of the pressures is indeterminate.
87. If the vertical through the centre of gravity falls within the base, the body is supported.
88. The stability increases with the base and decreases with the height.
89. If a body is on an inclined plane it will descend.

CHAPTER II.

THE MECHANICAL POWERS.

90. Object of machinery. Machines arranged under three classes; the *lever*, the *cord*, and the *inclined plane*.
§ I. 91. The lever.
92. Weight of the lever itself, a vertical force applied at the centre of gravity.
*93. General conditions of equilibrium as in art. 49.
*94. If the lever can only move in one plane, conditions as in art. 50.
95. In common applications the formula is

$$\Sigma . p q = 0.$$

96. Or more conveniently,

$$\Sigma . p r \sin . \alpha = 0.$$

97. If the forces act in parallel directions,

$$\Sigma . p r = 0.$$

98. If only two forces act on the lever, they are reciprocally proportional to their distances from the fulcrum.

99. Levers of three kinds.

100. Combination of levers.

§ II. 101. The balance—requisites of a good one.

102. *Horizontality:* centre of gravity in the perpendicular to the beam at the point of suspension.

103. *Sensibility:* formula for determining,

$$\tan.\ \theta = \frac{r\,w}{(2\,\mathrm{P} + \mathrm{W}).\,b}.$$

104. *Stability:* measured by the expression,

$$(2\,\mathrm{P} + \mathrm{W})\ b \sin.\ \theta.$$

105. The steelyard, a lever with a moveable weight.

106. The Danish steelyard, a lever with a moveable fulcrum.

§ III. 107. The wheel and axle.

* 108. Conditions of equilibrium as in art. 50.

109. In common applications,

$$\mathrm{P} : \mathrm{W} :: \text{rad. of axle} : \text{rad. of wheel}.$$

110. The thickness of the ropes must not always be neglected.

111. Pressure on the axis.

§ IV. 112. The cord, advantage of.

113. General condition of equilibrium, the forces if supposed to act on one point must be in equilibrio.

114. If only three forces act on the cord,

$$p' : p'' : p''' :: \sin.\ \alpha' : \sin.\ (\alpha + \alpha') : \sin.\ \alpha.$$

115. If the cord is fixed at its extremities, p' and p''' measure the reaction of the points or the tension of the cords.

116. If p'' is applied to a ring,

$$\alpha = \alpha', \text{ and } p'' = 2p \cos.\ \alpha.$$

§ V. 117. The pulley.

118. Conditions of equilibrium, the same as in 116.

119. The system where one is fixed and one moveable,

$$2P = W + w.$$

120. In the system, where there is a moveable *block* of pulleys,

$$nP = W + w.$$

121. In the system, where each has a separate rope,

$$2^n P = W + w' + 2w'' + 2^2 w''' + \&c.$$

122. In the system, where each rope is attached to the weight,

$$W = (2^n - 1)P + (2^{n-1} - 1)w' + (2^{n-2} - 1)w'' + \&c.$$

123. Pressures on the fixed points.

§ VI. 124. The inclined plane.

125. Conditions of equilibrium,

1st. The force must act in the vertical plane through the centre of gravity of the weight.

2nd. This vertical plane must be perpendicular to the intersection of the inclined and horizontal planes.

And

$$P : W : N :: \sin. BAC : \sin. PGE : \sin. PGH.$$

126. If the force is parallel to the plane,

$$P : W : N :: \text{height} : \text{length} : \text{base}.$$

127. If the force is parallel to the horizon,

$$P : W : N :: \text{height} : \text{base} : \text{length}.$$

128. In two inclined planes back to back,

$$W : W' :: L : L'$$

129. Case of a body sustained between two inclined planes.

§ VII. 130. The wedge,

131. Condition of equilibrium,

$$P : W + W' :: AB : AC + BC.$$

132. In the isosceles wedge,

$$P : 2W :: \text{sin. } \tfrac{1}{2} \text{ angle of wedge} : \text{radius.}$$

133. Consideration of the friction omitted, and pressure introduced instead of percussion.

§ VIII. 134. The screw.

135. Condition of equilibrium.

$$P : W :: h : 2\pi R.$$

CHAPTER III.

THE FUNICULAR POLYGON, AND CATENARY.

136. In the cord, the forces must be *divellent.*

137. Relation between the forces acting on a cord, deduced from 114.

138. When the forces are all parallel,

$$\frac{p''}{\text{cot. } \alpha + \text{cot. } \alpha'} = \frac{p'''}{\text{cot. } \beta + \text{cot. } \beta'} = \&c.$$

139. If weights act on the cord, the whole is in the vertical plane, and the horizontal tension is constant.

140. The directions of the reaction of the fixed points meet in the same point of the vertical through the centre of gravity.

141. The catenary.

142. The equation to the catenary, between s and x,

$$s = \sqrt{x^2 + 2cx}.$$

143. The equation between s and y,

$$y = c.\,h.\,l.\,\frac{s + \sqrt{c^2 + s^2}}{c}.$$

144. The equation between x and y,

$$y = c.\,h.\,l.\,\frac{x + c + \sqrt{x^2 + 2cx}}{c}.$$

DYNAMICS.

INTRODUCTION.

145. Motion accompanied with a certain *rate.*

146. *Velocity,* measured by the space that is or would be uniformly described in a given time; general equation,

$$v = \frac{ds}{dt}.$$

147. *Uniform force,* measured by the velocity generated in a given time; general equation,

$$f = \frac{dv}{dt}.$$

148. If f is required in terms of the space and time,

$$f = \frac{d^2 s}{dt^2}.$$

149. If v is required in terms of the force and space,

$$v dv = f ds.$$

SECTION I.

CHAP. I.

150. Method of investigation, as in Statics.

151. General formulæ for the force in each axis,

$$X = \frac{d^2 x}{d t^2}, \qquad Y = \frac{d^2 y}{d t^2}, \qquad Z = \frac{d^2 z}{d t^2}.$$

152. Formula to determine the velocity,

$$v^2 = 2\int(X\,dx + Y\,dy + Z\,dz) + C.$$

153. Velocity independent of the curve described between two points.

154. If the force tends to a fixed centre and is a function of the distance, the expression for the velocity is always integrable.

155. If the centre is not fixed, the expression is not an exact differential.

156. If the centre is fixed, the path is a *plane curve.*

157. Projections of the areas proportional to the time of description.

158. Reciprocally if the areas are proportional to the time, the force tends to a fixed centre.

159. If the motion is confined to a plane ∴ Z may be taken $= 0$.

160. If the motion is in a straight line ∴ Z may $= 0$ and also Y may $= 0$.

161. If $X = 0$, $Y = 0$ and $Z = 0$, uniform and rectilinear motion.

CHAPTER II.

CENTRAL FORCES.

162. Two equations are sufficient, as in 159.

163. Formulæ transformed to polar coordinates,

$$2\frac{dr}{d\omega}\cdot\frac{d\omega}{dt} + r.\frac{d^2\omega}{dt^2} = 0,$$

$$\frac{d^2r}{dt^2} - r.\frac{d\omega^2}{dt^2} + F = 0.$$

164. Areas proportional to the time of description.

165. *Angular velocity* varies inversely as square of distance.

166. Formula for the velocity,

$$v^2 = C - 2\int F\,dr.$$

167. Formulæ for the time,

$$dt = \frac{r^2 d\omega}{c} \quad \text{or} \quad \frac{dr^2}{dt^2} + \frac{c^2}{r^2} + 2\int F\,dr = C.$$

168. Formula for the equation to the orbit when the law of force is given, and vice versa;

$$c^2\left(\frac{du^2}{d\omega^2} + u^2\right) - 2\int F.\frac{du}{u^2} = C.$$

169. Formula for finding the law of force,

$$F = \frac{4a^2dp}{p^3dr}.$$

170. Expression for the velocity at any point,

$$v^2 = 2F.\tfrac{1}{4}\text{ chord of curvature through the centre.}$$

CHAPTER III.

RECTILINEAR MOTION.

171. *Absolute force;* $f = \varphi x^n$.
172. Initial motion.
173. Formula for the velocity generated by a variable force,

$$v^2 = C \pm 2 \int f dx.$$

174. Formula for the time,

$$t = \int \frac{dx}{v} + C.$$

175. Determination of the constants.
176. Formula for the velocity generated by an uniform force,

$$v = v' \pm ft.$$

177. Formula for the space in terms of the time,

$$s = c + v't \pm \tfrac{1}{2} f t^2.$$

178. If space and time have the same origin, $c = 0$.
179. If the point move from rest, $s = \frac{1}{2} f t^2$.
180. Formula for the velocity in terms of the space,

$$2fs = v^2 \backsim v'^2.$$

181. If the point move from rest, $v^2 = 2fs$.
182. In this case also, $s = \frac{1}{2} t v$.
183. Formulæ applicable to a *retarding force.*

CHAPTER IV.

CONSTRAINED MOTION.

184. Formula for velocity the same as in 152.
185. Velocity independent of the curve as in 153.
186. If the forces are in one plane, two equations are sufficient.
187. If the forces are all parallel, one is sufficient.
188. If the force is uniform, $v^2 = 2f(a - x)$.
189. Formula for the time as in 146.
190. If the force is uniform,

$$dt = \frac{ds}{\surd\left(2f(a-x)\right)}.$$

191. If we introduce polar coordinates,

$$v^2 = C - 2\int F\,dr,$$

$$dt = \frac{ds}{\surd(C - 2\int F\,dr)}.$$

*192. To determine the reaction of the curve, generally.
193. If the motion is in one plane,

$$N = Y\frac{dx}{ds} - X\frac{dy}{ds} + \frac{v^2}{\rho}.$$

193[a] If the forces are all parallel, one equation is sufficient.
194. If the force is uniform, $N = -f\frac{dy}{ds} + \frac{v^2}{\rho}$.
195. If the motion is in a straight line, $\rho = \infty$.

---

[a] Art. 193 has been accidentally repeated.

196. If we introduce polar coordinates,

$$N = F. \frac{r\,d\omega}{ds} + \frac{v^2}{\rho}.$$

197. Centrifugal force measured by $\frac{v^2}{\rho}$.

198. The same force exists whatever constrains the point.

199. Centrifugal force also $= \frac{4\pi^2 r}{T^2}$.

CHAPTER V.

THE MOTION OF A SYSTEM IN GENERAL.

200. Moving force measured by momentum generated in a given time.

201. *Centre of inertia;* formulæ as in 24.

202. D'Alembert's Principle. Impressed motions resolved into actual and destroyed.

203. The destroyed motions keep a system in equilibrio.

204. The impressed and actual motions if taken in contrary directions keep a system in equilibrio.

205. Considerations on the principle.

* CHAPTER VI.

THE MOTION OF ANY SYSTEM.

206. Formulæ for determining the motion,

$$\Sigma.\, m.\, \frac{d^2x}{dt^2} = \Sigma.\, m\,X, \qquad \Sigma.\, m.\, \frac{d^2y}{dt^2} = \Sigma.\, m\,Y,$$

$$\Sigma.\, m.\, \frac{d^2z}{dt^2} = \Sigma.\, m\,Z.$$

$$\Sigma . m \left(\frac{x d^2 y - y d^2 x}{dt^2}\right) = \Sigma . m (x \mathrm{Y} - y \mathrm{X}), \&c.$$

207. Motion of centre of inertia the same as if the whole mass were united in that point, and the resultant of all the moving forces were applied to it.

208. Principle of the conservation of the centre of inertia.

209. Principle of the conservation of areas.

210. The *invariable plane.*

211. To determine the position of the invariable plane at any instant.

212. Advantage of this plane in dynamical problems.

213. If the origin have an uniform rectilinear motion, the formulæ are of the same form as in 206.

214. The invariable plane continues parallel to itself during this motion.

215. Similarly, if the centre of inertia be assumed as the origin.

*CHAPTER VII.

THE MOTION OF A RIGID SYSTEM, PERFECTLY FREE.

216. Formulæ for the motion of a rigid system, the same as in 206, the sign $\int$ being introduced instead of Σ.

217. Formulæ retain the same form, when the centre of inertia is assumed as the origin. Motion of *translation,* and motion of *rotation.*

218. Motion round the centre of inertia.

*CHAPTER VIII.

THE MOTION OF A RIGID SYSTEM, WITH A FIXED POINT.

219. *Instantaneous* axis of rotation.
220. Position of this axis determined, at any instant.
221. Formulæ for the motion of rotation.
222. Formulæ, in a more convenient form,

$$A\,dp + (C - B).\,q\,r\,dt = N\,dt, \text{\&c.}$$

223. Assumption that $\int x'y'dm = 0$, $\int x'z'dm = 0$, $\int y'z'dm = 0$, gives the *principal* axes.
224. Equation for determining the principal axes.
225. Three principal axes and in general only one system.
226. *Moment of inertia* with reference to any axis.
227. To determine the moment of inertia with reference to any axis, when the moments with reference to the principal axes with the same origin are given.
228. Maximum and minimum moment of inertia.
229. Condition, when all the axes in the plane of x, y, are principal axes.
230. Condition when all the axes are principal axes.
231. To determine the moment of inertia with reference to any axis, when the moment with reference to a parallel axis through the centre of inertia is given.
232. Least of all the moments.

CHAPTER IX.

THE MOTION OF A RIGID SYSTEM ROUND A FIXED AXIS.

*233. Two fixed points equivalent to a fixed axis; discussion of general formulæ; expression for the angular velocity of rotation.

*234. Formula, when the force is uniform and parallel to the axis of x.

235. Investigation of the case, independent of the general formulæ; expression for the angular velocity,

$$\frac{d\omega}{dt} = \frac{\int r \mathrm{F} \cos.\delta . d m}{\int r^2 d m},$$

236. Modification of the formula.

*237. To determine the pressure on the axis arising from the inertia.

*238. If the axis be a principal axis through the centre of inertia, the motion is permanent.

*239. If the principal axis do not pass through the centre of inertia, the motion is not permanent, unless the origin be fixed.

*240. If the axis be not a principal axis, the motion will change.

*241. Condition of permanent motion.

*242. To determine the pressure on the axis arising from any instantaneous force.

*243. Case in which there is a single resultant pressure.

*244. Case in which there is a pressure parallel to the axis.

*245. If the direction of the force is perpendicular to the plane through the axis and the centre of inertia, there is no pressure on the axis.

*246. Centre of percussion.

SECTION II.

INTRODUCTION.

247. First law of motion

248. Force, impulsive or continual.
Note on the second law of motion.

CHAPTER I.

THE MOTION OF BODIES NEAR THE SURFACE OF THE EARTH.

249. Gravity, an uniform force, how measured.

250 Formulæ for the descent of a heavy body, from 176.

251. Formulæ for the ascent, &c.

252. Equation to the parabola described by a body projected obliquely,

$$y = x \tan. \epsilon - \frac{x^2}{4h \cos.^2 \epsilon}.$$

253 Range $= 2h \sin. 2\epsilon$,

254. Greatest height $= h \sin.^2 \epsilon$.

255. Time of flight $= 2\sqrt{\frac{2h}{g}}. \sin. \epsilon$.

256. Direction to hit a given mark.

257. Resistance of the air not considered.

258. Constrained motion. Velocity in all cases independent of the surface.

259. Motion down an inclined plane; accelerating force $= g \sin. \iota$

$$= g \cdot \frac{h}{l}.$$

261. Accelerating force of a body hanging freely and drawing another up an inclined plane is

$$\frac{p\,l - p'h}{l(p + p')} \cdot g.$$

262. Accelerating force of two bodies on two inclined planes with the same altitude is

$$\frac{(p\,l' - p'l)\,h}{ll'(p + p')} \cdot g.$$

263. Descent down a cycloidal arc with its axis vertical and base horizontal: time to the lowest point $= \pi \sqrt{\frac{a}{g}}$.

264. *Tautochronism* of the cycloid.

265. Motion up the opposite arc.

266. Time of descent down a circular arc.

267. If the arc is very small, the time is the same as down a cycloidal arc.

268. To make a body oscillate in a cycloid.

269. Pendulum, *Simple* and *Compound*. *Centre of Oscillation.*

270. To determine the centre of oscillation,

$$\therefore\ l = \frac{k^2 + a^2}{a}.$$

271. The axis of suspension and the axis of the centres of oscillation are reciprocal.

272. Variation of gravity.

273. To determine the motion of a body, when gravity cannot be considered constant.

CHAPTER II.

THE REVOLUTION OF THE HEAVENLY BODIES.

274. Phenomena of the heavenly bodies. Kepler's three laws.
275. Deduction from the first law; the force which acts on the planets tends to the centre of the sun.
276. Deduction from the second law; the force varies inversely as the square of the distance from the sun's centre.
277. Deduction from the third law; absolute force the same for all the planets.
278. Given the law of force, to determine the orbit, &c.
279. The same things being given, to determine the time.

CHAPTER III.

THE IMPACT OF BODIES.

280. Impact, *direct and central*; *direct and eccentric*; *oblique*.
281. Force of restitution; elasticity, perfect and imperfect; hardness and softness.
282. Formula for the velocity of two hard bodies after direct impact,

$$\therefore x = \frac{mv + m'v'}{m + m'}.$$

283. Formulæ for the loss and gain of two elastic bodies direct impact,

$$\text{loss of striking body} = (1 + e)(v - v')\frac{m'}{m + m'},$$

$$\text{gain of struck body} = (1 + e)(v - v')\frac{m}{m + m'}.$$

284. Case of elasticity, &c.
285. Relative velocity after impact is $e(v - v')$.
286. To determine the motion after oblique impact.
Formulæ for the motion after impact of a body on a hard plane,

$$e \tan. \rho = \tan. \iota; \quad v \sin. \iota = v' \sin. \rho.$$

287. Case of perfect elasticity.
288. Case of perfect hardness.
* 289. To determine the motion of two non-elastic bodies after eccentric impact.
290. To determine the motion when the bodies are elastic.
* 291. Case where the body struck is connected with a fixed axis.

ELEMENTS OF MECHANICS.

INTRODUCTION.

1. Mechanics is the science which treats of Equilibrium and Motion.

That branch of it which treats of Equilibrium is called *Statics*. That which treats of Motion is usually called *Dynamics*.

2. *Motion* is the passage from place to place. *Rest* is a permanency in the same place.

3. *Force* or *power* is the cause which produces or tends to produce motion or change of motion.

The consideration of the real nature of the cause, or of the force, does not belong to the present subject. We treat not of the cause but as producing, or tending to produce, certain *effects*.

4. The effect of a force (whether the effect be actual or virtual, i. e. whether it be actually produced, or partly or wholly counteracted) depends 1st. upon its intensity. 2ndly. Upon the point of application. 3rdly. Upon its direction. And it must be shewn how each of these is to be estimated.

1st. With respect to the intensity of the force, it must be observed, that all estimates of any quantity [illegible] reference to some fixed unit or standard. Hence, by assuming a force which is capable of producing some certain effect, as the unit of force, the intensity of any other force is expressed by the numerical ratio which its effect (whether actual or virtual) bears to the effect of the assumed unit of force. In the same manner also, by fixing on the linear unit to represent the unity of force, any other force is represented by the line, which bears to the linear unit the same ratio which the force in question bears to the unit of force.

2ndly. With respect to the point of application it may be observed, that as this point is a point in space it has its proper designation by reference to three coordinate axes or planes, as in analytic geometry.

3rdly. With respect to the direction of the force it may be observed, that it is the same as that of the line which a body acted on by that force would describe were it perfectly free. A line therefore will not only represent the magnitude of the force, but also its direction, and by reference to three coordinate axes the direction of any line is determined.

5. A body is an assemblage of material points. These material points, or elementary particles, are connected together in various ways according to the nature of the body.

In theoretic mechanics we commence by considering these points as perfectly isolated, without magnitude or weight, and as serving merely for the points of application of forces. We can then unite and connect these particles to form solid bodies, whether rigid or flexible. One class of bodies, where the connection between the particles seems less perfect, is referred to an almost distinct branch of the science.

The mechanics of *fluid* bodies, or more properly the equilibrium of fluids, would form the subject of a treatise on *Hydrostatics,* while their motions would come under *Hydrodynamics.*

STATICS.

SECTION I.

CHAP. I.

ON THE COMPOSITION AND EQUILIBRIUM OF FORCES APPLIED TO A POINT.

6. Prop. *Two equal forces applied to the same point in exactly opposite directions are in equilibrio.*

For no reason can be assigned why motion should ensue in the direction of one force which will not equally apply to the other.

7. Prop. *Any two forces applied to the same point, and forming any angle with the lines of their directions, cannot be in equilibrio.*

For suppose any two forces (fig. 1.), P and Q, acting on the point M, and forming the angle P M Q to keep it in equilibrio. Then, if a third force P′ is introduced, equal and opposite to P, motion must ensue in the direction of P′. But since P and P′ are equal and opposite they are in equilibrio (6) with each other, therefore motion will ensue in the direction of Q. Therefore the point must move in two directions at once, which is absurd.

8. *Cor.* Hence, since equilibrium cannot exist, motion will ensue, and the point will have a determinate velocity and direction as though it were the effect of a single force. This latter

deduce generally the several conditions of equilibrium of a point acted on by any forces.

13. Prop. *To find the resultant of any forces applied to the same point.*

The resultant is most conveniently obtained by resolving each component force according to three rectangular axes, and then by adding into one sum the forces which act in the same axis, and lastly by compounding these sums.

Thus let p', p'', p''', &c. be the forces, A (fig. 4.) their point of application, which is taken as the origin of the three rectangular axes A X, A Y, A Z.

Let α', β', γ', be the angles which p' makes with the axes,
α'', β'', γ'', those which p'' makes,
α''', β''', γ''', p''' &c.

In fig. 4. complete the rectangular parallelepiped taking A P representing any force as the diagonal.

Then A P may be resolved into A M, P n, n M,
or into A M, A N, A Q.

$$\begin{aligned} \text{And } AM &= AP \cos. PAM = p' \cos. \alpha', \\ AN &= AP \cos. PAN = p' \cos. \beta', \\ AQ &= AP \cos. PAQ = p' \cos. \gamma'. \end{aligned}$$

Proceeding similarly for the other forces and putting X, Y, Z, for the sum of the forces in each axis, we have

$$\begin{aligned} p' \cos. \alpha' + p'' \cos. \alpha'' + p''' \cos. \alpha''' + \&c. &= X, \\ p' \cos. \beta' + p'' \cos. \beta'' + p''' \cos. \beta''' + \&c. &= Y, \\ p' \cos. \gamma' + p'' \cos. \gamma'' + p''' \cos. \gamma'' + \&c. &= Z, \end{aligned}$$

or if Σ be used to denote the sum of the continued products of each force into the cosine of the angle, the above expressions will be

$$\Sigma . p \cos . \alpha = \mathrm{X} \qquad \Sigma . p . \cos . \beta = \mathrm{Y} \qquad \Sigma . p \cos . \gamma = \mathrm{Z}.$$

Therefore (12) the resultant

$$\mathrm{R} = \sqrt{(\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2)}.$$

And if A, B, Γ, be the angles which R makes with each axis, they may be found from the equations

$$\cos . \mathrm{A} = \frac{\mathrm{X}}{\mathrm{R}}, \ \cos . \mathrm{B} = \frac{\mathrm{Y}}{\mathrm{R}}, \ \cos . \Gamma = \frac{\mathrm{Z}}{\mathrm{R}}.$$

14. *Cor.* 1. If the forces are situated in the same plane

$$\therefore \mathrm{Z} = 0 \text{ and } \cos . \Gamma = 0. \quad \text{Hence } \mathrm{R} = \sqrt{(\mathrm{X}^2 + \mathrm{Y}^2)}$$

$$\text{and } \cos . \mathrm{A} = \frac{\mathrm{X}}{\mathrm{R}}, \ \cos . \mathrm{B} = \sin . \mathrm{A} = \frac{\mathrm{Y}}{\mathrm{R}} \ \therefore \tan . \mathrm{A} = \frac{\mathrm{Y}}{\mathrm{X}}.$$

15. In this case also we have

$$\mathrm{R} \cos . \mathrm{A} = \mathrm{X} = p' \cos . \alpha' + p'' \cos . \alpha'' + p''' \cos . \alpha''' + \&c.$$
$$\mathrm{R} \sin . \mathrm{A} = \mathrm{Y} = p' \sin . \alpha' + p'' \sin . \alpha'' + p''' \sin . \alpha''' + \&c.$$

Hence from any point S in the plane of the forces let fall perpendiculars upon the several directions of the forces, as for instance (fig. 5.) S M $= q'$ on A P′, S N $= q''$ on A P″, &c. and S O $= r$ on A R the resultant; join S A, and let S A $= s$, the ∠ S A X $= \theta$ and the ∠ P′ A X $= \alpha'$ as before.

$$\therefore \mathrm{SM} = \mathrm{SA} \ \sin . \ \mathrm{SAM}$$
$$\text{or } q' = s . \sin . (\theta - \alpha')$$

then multiplying the first of the equations X by s. sin. θ, and the second Y by s. cos. θ, and subtracting we have

$$\mathrm{R} s \, (\sin . \theta \cos . \mathrm{A} - \cos . \theta \sin . \mathrm{A}) =$$
$$p' s \, (\sin . \theta \cos . \alpha' - \cos . \theta \sin . \alpha') + p'' s \, (\sin . \theta \cos . \alpha'' - \cos . \theta \sin . \alpha'')$$
$$+ \ \&c.$$
$$\therefore \mathrm{R} s \sin . (\theta - \mathrm{A}) = p' s \sin . (\theta - \alpha') + p'' s \sin . (\theta - \alpha'')$$
$$+ \ \&c.$$
$$\therefore \mathrm{R} r = p' q' + p'' q'' + p''' q''' + \&c.$$

[illegible] measured by the perpendicular dis- [illegible] the moment [illegible] Hence the last equation [illegible] the moment of the resultant is equal to the sum of the moments of the several components.

By the sum of the moments we must understand the sum of each with its proper sign as determined by the value of [illegible] which shows that those are positive which tend to turn the point in one direction and those negative which tend to turn it in the contrary.

Cor. 2. If the forces are in the same straight line

$$\therefore Z = 0 \text{ and also } Y = 0.$$

$$\therefore R = X \text{ and cos. } A = 1.$$

∴ the resultant is parallel to or in the same line with axis of X.

17. If only two forces p', p'', making with each other an angle θ, act on the point, we have a very convenient form from the property of the triangle.

For (fig. 6.), $AD^2 = AB^2 + BD^2 - 2\,AB, BD \cos. B$. and $B = \pi - \theta$.

$$\therefore R^2 = p'^2 + p''^2 + 2\,p'\,p'' \cos. \theta.$$

also $$\sin. A = \sin. BAD = \frac{p''}{R} \sin. \theta$$

$$\sin. B = \sin. DAC = \frac{p'}{R} \sin. \theta.$$

18. Prop. *To determine the conditions of equilibrium of a* [illegible] acted on by any forces.

Here we must have $R = 0$ and $\therefore X^2 + Y^2 + Z^2 = 0$, [illegible]

and as each of these quantities is [illegible]

[illegible]

19. If a point acted on by any forces is supported on a curve, it will undergo a resistance arising from the impenetrability of the curve which must be perpendicular to the curve. For were it not so, this resistance may be resolved into two directions, one perpendicular to the curve, and the other in the direction of the tangent, of which the first alone can be effective in preventing the point from penetrating, the other would only affect the motion along the curve. Hence

Prop. To determine the conditions of equilibrium of a point sustained by any forces on a given curve.

Let N be the normal force of reaction,

α, β, γ, the angles which the normal makes with the axes of coordinates.

The resistance N may be considered as a new force acting upon the point, independently of the curve, and hence if X, Y, Z, are the sums of the other forces parallel to each axis respectively, and if N be similarly resolved, we have (18.)

$$\begin{aligned} N \cos. \alpha + X &= 0. \\ N \cos. \beta + Y &= 0. \\ N \cos. \gamma + Z &= 0. \end{aligned}$$

Hence, since $\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1$, we have.

$$N = \sqrt{(X^2 + Y^2 + Z^2)},$$

or the reaction is equal to the resultant and equilibrates all the other forces.

But since if α', β', γ', are the angles which the tangent makes with the axes

$$\cos. \alpha \cos. \alpha' + \cos. \beta \cos. \beta' + \cos. \gamma \cos. \gamma' = 0.^{b}$$

---

[b] This formula is deducible as a particular case of the following general problem. Given the angles which each of two straight lines makes with the axes, to find the angle at which the two lines are inclined to each other.

Let A B, and A C (fig. 7.) be any two lines through the origin parallel to

$$\text{and } \cos.\ \alpha' = \frac{dx}{ds},\ \cos.\ \beta' = \frac{dy}{ds},\ \cos.\ \gamma' = \frac{dz}{ds},$$

ds being the element of the curve,

$$\therefore \frac{dx}{ds} \cos.\ \alpha + \frac{dy}{ds} \cos.\ \beta + \frac{dz}{ds} \cos.\ \gamma = 0.$$

Therefore, multiplying the first of the three equations

by $\frac{dx}{ds}$, the second by $\frac{dy}{ds}$, and the third by $\frac{dz}{ds}$,

and adding these results

$$\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz = 0.$$

20. *Cor.* If the curve is in the vertical plane, two equations are sufficient, and

$$\mathrm{X}dx + \mathrm{Y}dy = 0$$

gives us an equation, by combining which with the equation to the curve, the position of the point may be determined if the forces are given.

the two lines, and let $\mathrm{AB} = \mathrm{AC} = 1$. Let A B make with the axes the angles α, β, γ, and let x, y, z, be the coordinates of B.

Similarly let A C make the angles α', β', γ', and x', y', z', be the coordinates of C. Since $\mathrm{AM} = \mathrm{AB} \cos.\ \alpha$, &c.

$$\therefore x = \cos.\ \alpha,\ y = \cos.\ \beta,\ z = \cos.\ \gamma$$
$$\text{and } x' = \cos.\ \alpha',\ y' = \cos.\ \beta',\ z' = \cos.\ \gamma'.$$

Join B, C $\therefore \mathrm{BC}^2 = \mathrm{AB}^2 + \mathrm{AC}^2 - 2\,\mathrm{AB}.\,\mathrm{AC} \cos.\ \mathrm{BAC}$

$$\therefore 2 \cos.\ \mathrm{BAC} \text{ or } 2 \cos.\ \theta = 1 + 1 - \mathrm{BC}^2$$
$$= 2 - \left((x - x')^2 + (y - y')^2 + (z - z')^2\right),$$

but $x^2 + y^2 + z^2 = 1$, &c.

$$\therefore \cos.\ \theta = xx' + yy' + zz'$$
$$= \cos.\ \alpha \cos.\ \alpha' + \cos.\ \beta \cos.\ \beta' + \cos.\ \gamma \cos.\ \gamma'.$$

Hence if the lines are at right angles to each other,

$$\cos.\ \alpha \cos.\ \alpha' + \cos.\ \beta \cos.\ \beta' + \cos.\ \gamma \cos.\ \gamma' = 0.$$

CHAPTER II.

ON THE COMPOSITION AND EQUILIBRIUM OF PARALLEL FORCES.

IN the last chapter we have discussed the composition of forces acting on the *same* point. We next proceed to consider forces acting on *different* points connected in some invariable manner, in order that we may ascend to the conditions of equilibrium of forces situated anywhere in space. For this purpose we will first examine the cases where forces are parallel to each other.

21. *PROP. To determine the resultant of two parallel forces acting on a rigid line without weight connecting their points of application.*

Let p' and p'' be two forces in any direction acting on the points A and B connected by the invariable line AB. (fig. 8.) Then we shall not at all disturb the state of the system if we suppose the two forces applied at the point O, in which the directions of the forces meet when produced.

If then α', α'', and A are the angles which the directions of the forces and the resultant R, make with the axis of x respectively, we have (15)

$$\begin{aligned} R \cos. A &= p' \cos. \alpha' + p'' \cos. \alpha'' \\ R \sin. A &= p' \sin. \alpha' + p'' \sin. \alpha'' \end{aligned}$$

and also $Rr = p'q' + p''q''$

These expressions hold good, if we suppose the directions of p' and p'' to vary. Hence if $\alpha' = \alpha''$ or the forces become parallel and in the same direction, we find, 1st. $A = \alpha' = \alpha''$; or resultant is parallel to components.

2ndly. $R = p' + p''$; or resultant = the sum of components.

3rdly. If the point from which the perpendiculars are let fall be taken in the direction of the resultant,

$$\therefore\ r = 0\ \therefore\ p'q' + p''q'' = 0.$$

Hence if p' and p'' have the same sign, q' and q'' must be in opposite directions, and they are also in the same straight line.

Hence $p' : p'' :: q'' : q'$
$:: GN : GM$ (fig. 9.)
$:: BG : GA$, by sim. triangles.

But if the forces act in contrary directions we have still the resultant parallel to the components; but $R = p' - p''$, or resultant = the difference of components, also the perpendiculars must fall on the same side of the resultant, and the point G will fall in the line AB produced in the direction of the greater force (fig. 9 *.) and

$$p' : p'' :: BG : GA.$$

But if $p' = p''$ $\therefore$ $R = 0$, also since $BG = AB, \dfrac{p'}{p'-p''}$ we have here $BG = \infty$ $\therefore$ no single force can equilibrate two equal forces in contrary but not opposite directions.

In this case there is a tendency to produce rotatory motion.

22. *Cor.* 1. If we have three parallel forces, we can first find the magnitude and point of application of the resultant of two, and then compound this resultant with the third, and so on. But if any of the forces have a *contrary* direction, as q', q'', q''', &c., then we must first find the point of application of the resultant of p', p'', p''', &c., which act in one direction, and then the point of application of the resultant of q', q'', q''', &c., and then proceed as in the latter part of the proposition.

23. *Cor.* 2. If the forces p' and p'' turn round their points of application, their intensities and parallelism still remaining, the magnitude of the resultant and its point of application will also continue unchanged. And similarly for any number of forces.

Hence in every system of points connected in an invariable manner and acted upon by parallel forces, there is a certain point through which the resultant of all the forces passes, whatever be the position of the system. This point is called the *Centre of Parallel Forces.*

24. **Prop.** *To find the position of the centre of any number of parallel forces, the coordinates of the points of application of the forces being given.*

Let the points P′, P″, P‴, &c. be acted on by the parallel forces p', p'', p''', &c. and referring the system to three rectangular axes, let x', y', z', x'', y'', z'', &c. be the coordinates of the points P′, P″, &c.

1st. Taking only two points P′. P″, whose ordinates on AX (fig. 10.) are $AM' = x'$, and $AM'' = x''$, if G be the centre of the two forces, and its ordinate be $AN = \xi$, then we have

$$p' : p'' :: P''G : GP' :: M''N : NM'$$
$$\therefore p', M'N = p'', M''N$$
$$\text{or, } p' (\xi - x') = p'' (x'' - \xi).$$
$$\therefore (p' + p'')\, \xi = p'x' + p''x''.$$

Similarly if the point G and a third point P‴ be joined it may be easily shown that, if ξ'' be the coordinate of G″ the centre of the forces $p' + p''$, and p'''

$$(p' + p'' + p''')\, \xi'' = p'x' + p''x'' + p'''x'''$$

So generally, if $x_,$ $y_,$ $z_,$ be the coordinates of the centre of all the forces

$$x_{,} = \frac{p'x' + p''x'' + p'''x''' + \&c.}{p' + p'' + p''' + \&c.}$$

$$y_{,} = \frac{p'y' + p''y'' + p'''y''' + \&c.}{p' + p'' + p''' + \&c.}$$

$$z_{,} = \frac{p'z' + p''z'' + p'''z''' + \&c.}{p' + p'' + p''' + \&c.}$$

$$\text{or } x_{,} = \frac{\Sigma.\, px}{\Sigma.\, p} \qquad y_{,} = \frac{\Sigma.\, py}{\Sigma.\, p} \qquad z_{,} = \frac{\Sigma.\, pz}{\Sigma.\, p}.$$

The sign Σ denoting as before the sum of the series (14).

25. *Cor.* 1. If the points are all in the same plane, this may be taken for one of the coordinate planes, as XAY.

In this case $z' = z'' = z'''$ &c. $= 0$. $\quad \therefore\ z_{,} = 0$.

26. *Cor.* 2. If the points are all in one straight line, this may be taken for one of the axes, as AX.

In this case $z_{,} = 0$, and $y_{,} = 0$.

27. The intensity of any force multiplied by the perpendicular, let fall from the point of application upon a plane, is called *the Moment of the force with reference to the plane.*

Thus $p'x'$, $p''x''$, &c. are the moments of p', p'', &c. with reference to the plane YAZ.

28. *Cor.* 3. If R be the resultant of all the parallel forces

$R = p' + p'' + p''' +$ &c. with their proper signs.
Also, $Rx_{,}$, $= p'x' + p''x'' + p'''x''' +$ &c.

Hence the moment of the resultant with respect to any plane is equal to the sums of the moments of the components.

The same result would also obtain, if the origin were changed.

29. Prop. *To determine the conditions of equilibrium of any number of parallel forces.*

1st. We must have $R = 0 \therefore p' + p'' + p''' + \&c. = 0$.

2ndly. Since the forces are parallel we may assume the axis of z parallel to their directions, and since we must also have

$$p'x' + p''x'' + p'''x''' + \&c. = 0.$$
$$p'y' + p''y'' + p'''y''' + \&c. = 0.$$

in order that x, and y, may not be ∞.

Hence the necessary conditions of equilibrium.

1st. *The sum of the forces must be equal to zero.*

2ndly. *The snm of their moments with reference to two planes parallel to their directions, must also each be equal to zero.*

If any of the forces act in contrary directions, they must of course be considered as negative; but if the sum of the negative forces is equal to the sum of the positive ones, and their respective resultants are not opposite as well as contrary, there can be no common resultant and no equilibrium.

30. *Cor.* 1. If the equilibrium subsist, one force as p' must be equal and opposite to the resultant of all the others.

31. *Cor.* 2. If the equilibrium subsist with one direction, it will subsist with all directions of the parallel forces.

CHAPTER III.

ON THE COMPOSITION AND EQUILIBRIUM OF FORCES APPLIED AT DIFFERENT POINTS OF AN INVARIABLE SYSTEM, AND ACTING IN ANY DIRECTION.

32. Prop. *To investigate the effects of any forces acting on different points connected in an invariable manner.*

Let p', p'', p''', &c. be the forces anywhere situated in space, then referring to three coordinate planes. Let

x', y' z',	be the coordinates of the point of application of p'.
α', β', γ',	the angles made by the direction of p' with parallels to the three axes.
x'', y'', z'',	the coordinates of p''.
α'', β'', γ'',	the angles of p'', &c.

Each of these forces may be resolved into three, parallel to the three axes, and thus three sets of parallel forces may be obtained. By this resolution we obtain

$$\begin{aligned}
p' \cos. \alpha' + p'' \cos. \alpha'' + p''' \cos. \alpha''' + \&c. &= X. \\
p' \cos. \beta' + p'' \cos. \beta'' + p''' \cos. \beta''' + \&c. &= Y. \\
p' \cos. \gamma' + p'' \cos. \gamma'' + p''' \cos. \gamma''' + \&c. &= Z.
\end{aligned}$$

X, Y, Z, representing the resultants of each set of parallel forces (28.).

But since each of the first set of these forces is parallel to AX, its moment with reference to the plane Y A Z is necessarily $= 0$, and we have therefore the sums of the moments with reference to X A Z, and X A Y (29) viz.

$p' \cos. \alpha'. y' + p'' \cos. \alpha''. y'' + p''' \cos. \alpha'''. y''' + \&c.$

with reference to X A Z, and parallel to X A Y.

$p' \cos. \alpha'. z' + p'' \cos. \alpha''. z'' + \&c.$

with reference to X A Y, and parallel to X A Z.

So for the other sets,

$p' \cos. \beta'. x' + p'' \cos. \beta''. x'' + \&c.$

with reference to Y A Z, and parallel to X A Y.

$p' \cos. \beta'. z' + p'' \cos. \beta''. z'' + \&c.$

with reference to X A Y, and parallel to Y A Z.

and,

$p' \cos. \gamma'. x' + p'' \cos. \gamma''. x'' + \&c.$

with reference to Y A Z, and parallel to X A Z.

$p' \cos. \gamma'. y' + p'' \cos. \gamma''. y'' + \&c.$

with reference to X A Z, and parallel to Y A Z.

We have here two sets of moments parallel to each plane, but we may observe, that each of the two sets *tend to turn the system*[a] in contrary directions; accordingly, collecting them together with their proper signs, we obtain for the moments parallel to the three planes

$$p'(\cos. \alpha'. y' - \cos. \beta'. x) + p''(\cos. \alpha''. y'' - \cos. \beta''. x'') + \&c. = \mathrm{L}.$$
$$p'(\cos. \gamma'. x' - \cos. \alpha'. z) + p''(\cos. \gamma''. x'' - \cos. \alpha''. z'') + \&c. = \mathrm{M}.$$
$$p'(\cos. \beta . z' - \cos. \gamma'. y) + p''(\cos. \beta'. z'' - \cos. \gamma''. y'') + \&c. = \mathrm{N}.$$

---

[a] Figure 11 may probably represent the tendency of two moments parallel to the plane X A Y, or the plane of the paper, to turn the system in contrary directions.

Or the expressions may be thus written:

$$\Sigma.\, p \cos.\, \alpha = \mathrm{X} \qquad \Sigma.\, p \cos.\, \beta = \mathrm{Y} \qquad \Sigma.\, p \cos.\, \gamma = \mathrm{Z}.$$

$$\Sigma.\, p\, (\cos.\, \alpha.\, y - \cos.\, \beta.\, x) = \mathrm{L}.$$
$$\Sigma.\, p\, (\cos.\, \gamma.\, x - \cos.\, \alpha.\, z) = \mathrm{M}.$$
$$\Sigma.\, p\, (\cos.\, \beta.\, z - \cos.\, \gamma.\, y) = \mathrm{N}.$$

We then have the six equations, X, Y, Z, L, M, N, to determine the effects of the forces; and we will proceed

33. *Prop. To determine the conditions of equilibrium.*

By combining the conditions of equilibrium of each set of parallel forces, we find that each of the six equations must separately equal zero (29);

$$\mathrm{X} = 0,\ \mathrm{Y} = 0,\ \mathrm{Z} = 0, \text{ and also, } \mathrm{L} = 0,\ \mathrm{M} = 0,\ \mathrm{N} = 0.$$

34. *Cor.* 1. The equations of equilibrium on any particular supposition are easily found by modifying these six equations.

Thus if the forces met in a point, this point may be taken as the origin, and then the coordinates of the points of application being all zero, the three equations, L, M, N, are necessarily fulfilled; and there remain the other three as in art. 13.

35. *Cor.* 2. If the forces are parallel, as, for instance, to Ax we have the equations Y, Z, and N, necessarily = 0, and there remain the other three as in art. 29.

36. *Cor.* 3. If the forces are in the same plane, as, for instance, X A Y,

then since z', z'', z''', &c. $= 0$ and $\cos.\, \gamma = 0$, &c.

the equations Z, M, N are necessarily fulfilled, whence *the conditions of equilibrium for the same plane are, that the sums of the forces resolved in directions parallel to two axes should each* = 0, *and that the sum of the moments round the third should also* = 0.

37. Prop. *To determine the conditions that there may be a single resulting force.*

Assuming that the quantities X, Y, Z are not equal to zero, we have to compound three forces respectively parallel to three axes: and we cannot have a single resultant unless the three meet in a point. And as any point taken indifferently in the line of direction of a force may be considered the point of application of the force: let $x_{,}$, $y_{,}$, $z_{,}$ be the coordinates of the point, hence we have (28)

$$X y_{,} = p' \cos.\ \alpha'.\ y' + p'' \cos.\ \alpha''.\ y'' + \&c.$$
$$X z_{,} = p' \cos.\ \alpha'.\ z' + p'' \cos.\ \alpha''.\ z'' + \&c.$$

Similarly for the others.

$$\text{Hence,} \quad \begin{aligned} X y_{,} - Y x_{,} &= L. \\ Z x_{,} - X z_{,} &= M. \\ Y z_{,} - Z y_{,} &= N. \end{aligned}$$

Whence multiplying the first by Z, the second by Y, and the third by X, and adding

$$LZ + MY + NX = 0.$$

is the equation of condition for a single resultant.

38. *Cor.* 1. The magnitude of the resultant will be as in art. 13.

$$R = \sqrt{(X^2 + Y^2 + Z^2)}\ \&c.$$

39. If at the same time $X = 0$, $Y = 0$, $Z = 0$, the above equation of condition is satisfied, but yet there *may not* be a single resultant. We have seen in art. 21, that if the resultant of any parallel forces $= 0$, the forces may be reduced to a pair of forces equal and contrary, but not opposite. So here each of the sets may be reduced to a pair, which may be compounded easily into a single pair.

40. If X = 0, and Y = 0, but Z do not = 0, then the equation is satisfied if L = 0. Hence the forces in the plane X A Y are in equilibrio (36), and there remain only the forces parallel to the axis of Z. But if L do not = 0, then the forces in the plane X A Y are reducible to a pair, which, if combined with the force Z, give us two forces in different planes, and consequently no single resultant.

41. If only X = 0, the pair thus presented, if combined with the forces Y and Z, give us a single resultant.

42. If at the same time L = 0, M = 0, N = 0, but one or more of the forces X, Y, Z not = 0, the equation of condition is satisfied, and we have a single resultant passing through the origin (see 34).

43. If one or more of the moments L, M, N, and one or more of the forces X, Y, Z, are not = 0, and if the equation of condition is satisfied, we have a single resultant which does not pass through the origin.

44. But if the equation is not satisfied, the system cannot be reduced to a single force.

The symmetry of the equation of condition shews that the last articles apply as well to any combinations of L, M, N, X, Y, Z, as to the combinations specified.

CHAPTER IV.

ON THE CONDITIONS OF EQUILIBRIUM, &c. WHEN THERE ARE ANY FIXED POINTS IN THE SYSTEM.

45. When a force is applied to a fixed point it is destroyed; or, more properly, another force is brought into action which keeps it in equilibrio.

46. When a force is applied to a point which is constrained to move in a certain line, if the direction of the force is perpendicular to the line, there is no reason why the point should move in one direction along the line rather than the other. Hence no motion will ensue. If the force is oblique, it may be resolved into two; one perpendicular wholly counteracted, the other parallel to the line wholly effective.

47. When a force is applied to a point situated on an impenetrable plane, similarly if the force is perpendicular it is counteracted, but if oblique it may be resolved as above.

The new force thus brought into action is called a force of resistance or reaction.

48. With respect to its direction, it appears that a fixed point is capable of resistance in any line passing through it. That a line is capable of resistance in any line perpendicular to it, and that consequently its resistance is confined to a plane perpendicular to the line at the point where the force acts. That the resistance of a surface is confined to a line perpendicular to the surface at that point.

49. *Prop. To determine the conditions of equilibrium when there is a fixed point in the system.*

Here it is necessary and sufficient that the forces should be reducible to a single resultant passing through the fixed point.

Therefore (43) we must have $L = 0$, $M = 0$, and $N = 0$, or the forces must have no resulting moment relative to that point.

The resistance may be estimated from X, Y, Z, and the *pressure on* the point is the resultant of these forces.

50. Prop. *To determine the conditions of equilibrium when there are two fixed points in the system.*

Here the line by which those points are connected, and therefore every point in it is fixed. It becomes therefore as a fixed axis. Suppose then this axis as the axis of Z, we at once satisfy $Z = 0$, $M = 0$, and $N = 0$ (36). As to the remaining forces, X, Y, their resultant must pass through the origin which is a fixed point as being on the axis, and since M and N are necessarily satisfied, the equation $L = 0$ gives the condition of equilibrium, which must be satisfied independently of the resistances.

51. *Cor.* If the system be capable of sliding along the axis, we must introduce $Z = 0$ as another condition of equilibrium.

52. If there are *three* fixed points in the system not in the same right line, the whole system is immoveable and equilibrium necessarily exists.

53. Prop. *To find the conditions of equilibrium if a point is situated on a plane.*

Here the forces must be reducible to a single resultant directed perpendicularly against the plane at the point of contact. Taking therefore this point as the origin, and the plane as the plane X A Y, it is requisite that the equations

$$X = 0,\quad Y = 0,\quad L = 0,\quad M = 0,\quad N = 0,$$

should be satisfied (43). There then remains only a single resultant, which (if it is *towards* the plane) is necessarily counteracted by the resistance.

If it is *from* the plane no resistance is excited, and the six equations must be satisfied for equilibrium.

54. If two points are on the plane, the forces of resistance are two parallel forces perpendicular to the plane at the points of contact, and these two may (21) be reduced to a parallel resultant applied at some intermediate point.

55. If there are more points than one on the plane in the same right line, all but the extreme points are useless to establish equilibrium.

56. If there are three points on a plane not in the same right line, there are three forces of resistance excited, with directions perpendicular to the plane. These may be reduced to a parallel resultant applied at some point within the triangle formed by joining the points.

57. We might next proceed to investigate the pressures on the points of support; but as problems on this subject seldom involve more than the action of parallel forces, and are perhaps more conveniently solved without having recourse to the general formulæ, we have considered such a discussion foreign to an elementary treatise. Some of the principal properties of the pressures on fixed points will be investigated in the application of these principles to heavy bodies.

CHAPTER V.

ON SYSTEMS OF VARIABLE FORM.

58. The conditions of equilibrium of a *rigid* system are not sufficient in a system of a *variable* form. In these, besides a progressive and rotatory motion, various other motions may have a tendency to arise according to the nature of the system. The general investigation of the principles of equilibrium in such systems would exceed the limits of an elementary treatise; especially as for practical purposes, the consideration of certain particular systems may suffice. Such investigations have therefore, as not being general, been deferred. See Section II. chap. 5.

STATICS.

SECTION II.

CHAP. I.

ON THE CENTRE OF GRAVITY.

59. In the preceding section we have discussed certain theorems relating to forces applied to a point or system of points abstractedly. We proceed now to apply those deductions to heavy bodies.

Experiment shews that every particle of matter is subject to a force which attracts it in a direction perpendicular to a horizontal plane or the surface of stagnant water. In reality the directions of the forces acting on several particles meet nearly in the centre of the earth: but as this centre is very distant compared with the distance of any particles considered together, we may, without any sensible error, consider them parallel.

This force is called *gravity*.

60. Experiment also shews, that the intensity of gravity varies in different parts of the earth's surface; that it is least at the equator, and increases towards the pole in proportion to the square of the sine of the latitude. It shews also that in the same latitude, the intensity varies at different points in the same vertical line; that it varies inversely as the square of the distance from the centre: but for any points in the same system, or

any bodies nearly in the same place, this variation of intensity, as well as the variation of direction may be neglected without error.

61. We shall consider, therefore, a heavy body as an assemblage of material points acted on by equal and parallel forces in the direction of the vertical to the earth's surface.

All these forces have a resultant equal to their sum (28) and parallel to their common direction. This resultant is what is called the *weight* of the body.

Hence the weight of an *homogeneous* body (or a body of the same uniform materials) is independent of its form, and is also proportional to its volume.

62. In heterogeneous bodies we have different weights under the same volume, and that body is called the more *dense* which has the greater number of particles. The relation, then, between the number of particles, or weight, of a body and its volume, or bulk, is the density. Assuming then a given volume of any substance as the unit of density, the densities of any other substances may be determined with reference to it, from the equation $D = \frac{W}{V}$. D, W, V representing respectively the density, weight, and volume.

63. When we transfer a body from one place to another on the surface of the earth, its weight varies in proportion to its gravity, although it is impossible to ascertain the difference by means of a balance, because both the weight and the body are changed. The equation, therefore, $W = DV$, is only true for the same place: but it may be made general by introducing g to denote the relation between gravity at any other place and some place where it is assumed as unity. We have then generally $W = g\,D\,V$.

64. Since, then, all the points of a heavy body are solicited by parallel forces, it follows (23) that if the body is made to

take successively several positions, the resultant of the forces will always pass through a certain point in the body. This point we called generally the *centre of parallel forces* (23); we call it when treating of bodies the *centre of gravity*.

65. Prop. *To find the centre of gravity of any body.*

As a body is considered to be an assemblage of material particles, we have a system of points acted on by gravity in a vertical direction. The weights, therefore, of each particle form a system of parallel forces. And if p', p'', p''', &c. represent the weights of the particles which may be supposed collected in their respective centres of gravity; x', y', z'; x'', y'', z'', &c. their coordinates, we have for the coordinates x, y, z, of the centre of gravity of the whole body (24).

$$x' = \frac{p'x' + p''x'' + p'''x''' + \&c.}{p' + p'' + p''' + \&c.}$$

$$y' = \frac{p'y' + p''y'' + p'''y''' + \&c.}{p' + p'' + p''' + \&c.}$$

$$z' = \frac{p'z' + p''z'' + p'''z''' + \&c.}{p' + p'' + p''' + \&c.}$$

If the body is homogeneous, the weights of each part or particle are simply proportional to the volumes (61); and if v', v'', v''', &c. denote the volumes of the particles, and V the whole volume, we have

$$x_{,} = \frac{v'\,x' + v''\,x'' + v'''\,x''' + \&c.}{V}$$

$$y_{,} = \frac{v'\,y' + v''\,y'' + v'''\,y''' + \&c.}{V}$$

$$z_{,} = \frac{v'\,z' + v''\,z'' + v'''\,z''' + \&c.}{V}$$

Hence we obtain *that the whole volume multiplied by the distance of its centre of gravity from a plane is equal to the sum of each part of the volume into the distance of its centre of gravity from the plane.*

66. The above expressions may be made convenient for computation by the introduction of the integral calculus, the principles of which give us

$$x_{,} = \frac{\int x dv}{\int dv}. \qquad y_{,} = \frac{\int y dv}{\int dv}. \qquad z_{,} = \frac{\int z dv}{\int dv}.$$

In which expressions we may observe, that x, y, z denote the distances of the centre of gravity of dv from the planes.

By means of these three equations the exact position of the centre of gravity may be determined: but the three equations are not always necessary.

67. For instance, if *the centre of gravity of a plane figure* be required, two equations are sufficient, since the centre of gravity must be in that plane.

Thus for a plane curve, if ds the differential of the arc $= dv$,

$$x_{,} = \frac{\int x ds}{s}. \qquad y_{,} = \frac{\int y ds}{s}.$$

If *the arc be symmetrical on the axis of x,* the centre of gravity is in the axis, and

$$y_{,} = 0 \ \therefore \text{ one equation } x_{,} = \frac{\int x ds}{s}$$

is sufficient.

But if it be a curve of double curvature we must refer it to three axes.

68. *For an area,* since $dv = y dx$, we have

$$x_{,} = \frac{\int x y dx}{\int y dx}. \qquad y_{,} = \frac{\int y^2 dx}{\int y dx}.$$

If the *area is symmetrical* on the axis of x, the one equation

$$x_{,} = \frac{\int xydx}{\int ydx}.$$ is sufficient.

69. *For a surface of revolution round the axis of x* where $dv = 2\pi\, yds$ the one equation $x_{,} = \frac{\int xyds}{\int yds}$ is sufficient.

70. *For a solid of revolution* where $dv = \pi\, y^2dx$ similarly

$$x_{,} = \frac{\int y^2xdx}{\int y^2dx}$$ is sufficient.

By the proper substitutions for dv in the general equations we may find the centre of gravity in any other cases.

71. The centre of gravity may be found in many cases by more simple methods.

Thus the centre of gravity of an uniform straight line is in its middle point.

72. The centre of gravity of a parallelogram is in the intersection of the straight lines which bisect the opposite sides.

For the parallelogram A B C D (fig. 12.) may be supposed to be made up of straight lines parallel to A B, which have their centres of gravity in the line E F bisecting A B and D C, whence the centre of gravity of the parallelogram must be in E F, similarly it will be in H K, bisecting A D and B C, therefore it is in G their intersection.

73. The centre of gravity of the triangle may be found similarly. Supposing the triangle similarly made up, the centre of gravity of A B C (fig. 13.) will be in B E, which is drawn from B bisecting A C, and also in C F drawn from C bisecting A B therefore it is in G their intersection; and as by similar triangles,

$$BG : GE :: BC : FE$$
$$:: 2 : 1$$
$$BG = \frac{2}{3} BE.$$

The same principles may be extended to other simple figures and bodies.

74. The characteristic property of the centre of gravity is, that if it is supported, the body will remain at rest in all positions. This is evident, because the resultant of the forces will pass through a fixed point.

75. If there is any other fixed point in the body, it is necessary and sufficient for equilibrium that the centre of gravity should be in the vertical line passing through the fixed point. For since the weight is a vertical force applied to the centre of gravity, it may be considered as applied to the fixed point when its direction passes through it, and therefore it will be then and then only counteracted.

Hence the centre of gravity must be either vertically above or vertically below the fixed point or point of suspension in case of equilibrium.

76. If the equilibrium is such, that the body if disturbed tends to return to its original position, it is called *stable* equilibrium. If the body tends to remove farther from its position, it is called *unstable.* If it remains at rest in any position we have *neutral* equilibrium.

77. If the centre of gravity is vertically below the point of suspension, and any disturbance were to move it from its position, as in fig. 14, we have two forces acting on the body, the resistance of the point of suspension in the direction of G S, and its weight in the direction of G K, the resultant of which falls within the angle S G K and will not vanish until G K coincides with G H, we have here then *stable* equilibrium.

If G coincide with S we have *neutral* equilibrium.

If G be vertically above S we have *unstable* equilibrium, see fig. 15.

78. Hence we derive a practical method of finding the centre of gravity. Suspend the body by two different points successively, and the intersection of the vertical or plumb-lines in each case is the centre of gravity.

79. The *pressure* on the point, by which a body is suspended, is clearly in the case of equilibrium equal to the weight of the body.

80. If a body is suspended from two points it will not rest until the centre of gravity is brought to the vertical plane passing through the two points of suspension.

The *pressures* on the two points may be determined by observing that the weight of the body may be considered as applied to the centre of gravity, and that it must be resolved into two forces whose directions must pass through the fixed points in order to be counteracted. Hence the two lines drawn from the centre of gravity to the points of suspension are in the directions of the pressures, and the vertical is the direction of the weight; and the weight is given. Hence in a parallelogram of forces, we have the two sides and the diagonal given in position and one given in magnitude, whence the magnitude of the other two are determined.

81. If there are three fixed points, the body is fixed necessarily. With respect to the *pressures* on each point, we have the three lines drawn from the centre of gravity to the points in their direction, and the vertical is the direction of the weight, the magnitude of which is given. Hence we have in a parallelepiped the three sides and diagonal given in position, and one given in magnitude, to determine the magnitude of the other three as before.

82. If a body be placed on a horizontal plane, and touch it

only in one point, it will be in equilibrio if the weight be counteracted by the resistance of the plane, or if the vertical through the centre of gravity and the perpendicular to the plane at the point of contact coincide. In this case the smallest force will disturb the body, but the equilibrium may be *stable*, *unstable*, or *neutral*.

83. If a body touch the plane in two points, the vertical through the centre of gravity and the perpendiculars to the plane at the points must be in the same plane if the body is in equilibrio. Thus if A B C (fig. 16.) represent a vertical section through the two points of support P, Q, there will not be an equilibrium unless G H the perpendicular from the centre of gravity is in the same plane, or, which is the same thing, meets the line P Q.

The pressures on P and Q may be estimated by the theory of parallel forces, and

$$P : Q : W :: HQ : HP : PQ.$$

P, Q, W, representing the pressures on the two points and the weight of the body respectively.

84. If there are more than two points of contact in the same right line, only the extreme points are requisite for equilibrium, and the problem of the pressures is indeterminate, as the pressures may be any how distributed; a third point may support nothing or any portion of the weight.

85. If a body touch the plane in three points, not in the same right line, it will be supported if the vertical through the centre of gravity fall within the triangle formed by joining the three points.

To estimate the pressures in this case, suppose P Q R (17.) to be the triangle formed by joining the three points, H the point where the vertical through G meets it we have here to resolve the weight or vertical force at H into three others parallel to it, acting at P, Q, and R; join P H and produce it to meet R Q in K. Then by resolving W at H, by parallel forces we have

P : W :: H K : P K :: triangle R H Q : triangle P Q R.

So joining Q H and producing

Q : W :: H M : Q M :: triangle P H R : triangle P Q R

and similarly

P : Q : R : W :: triangle R H Q : triangle P H R : triangle P H Q : triangle P Q R.

86. If the body touches the plane in more points than three, the pressure on the several points is indeterminate, but the sum of the pressures is equal to the weight of the body.

87. If the vertical through the centre of gravity of a body on a plane meets the plane in a point within the base, the body will be supported. For the resultant of the parallel forces of resistance must fall within the figure formed by joining the several points of contact, since the forces are in the same direction (21) and consequently can counteract the weight.

If the vertical falls without the base, we have two parallel forces in contrary but not opposite directions and the body will turn over.

88. The stability of the body will depend on the excess of the shortest line that can be drawn from the centre of gravity to the perimeter of the base above the vertical from the centre to the plane, on the excess of G P above G H, in fig. 18. For G must be raised to an elevation equal to this difference in order to turn the body over the edge of the base at P.

Hence the greater the base H P, the greater the stability, if the height of G is the same. And the greater H G the less the stability, if H P is the same.

89. If the plane be not horizontal but inclined the body will descend, if there is no restraint from friction.

For the weight of the body may be resolved into two forces, one perpendicular to the plane and the other parallel to it, of which the perpendicular can alone be counteracted by the plane. In this case if the vertical through the centre of gravity fall within the base of the body, the body will slide: if it fall without, the body will roll or tumble over.

These principles might be extended to the investigation of the circumstances of equilibrium upon a point and a surface, or upon two surfaces, &c. but enough has been said to develop the general principles of such investigations.

CHAPTER II.

ON THE MECHANICAL POWERS.

The object of machinery is to enable us to transmit and to economize certain forces at our disposal, in order to counteract some other forces.

The most simple machines are generally called the mechanical powers, and they may be most conveniently classed under three, viz. the *lever,* the *cord,* and the *inclined plane.*

The *first* class comprehends every machine consisting of a solid body capable of revolving on an axis, as the *balance,* and the *wheel* and *axle.*

The *second* class comprehends every machine in which force is transmitted by means of flexible threads, cords, &c., hence it includes the *pulley.*

The *third* comprehends every machine in which a hard surface inclined to the horizon is introduced, as in the *wedge* and the *screw.*

In machines the force which is used to sustain or overcome any opposition is frequently called the *power;* the opposition to be overcome is called the *weight.* But the terms *power* and *weight* must not be understood as implying any necessary distinction between the forces.

The advantage of machines consists in the introduction of certain *fixed points* or *props,* by the reaction of which a new force is introduced.

CHAPTER II. §. I.

THE LEVER.

91. A *lever* is an inflexible bar capable of angular motion round a point, which is called the *fulcrum*.

The bar may be *straight* or any how *bent*.

92. The lever is generally considered at first as being without weight, but evidently the weight is merely another force applied at the centre of gravity in a direction perpendicular to the horizon. And in order to obtain the properties of the lever we will commence with the following general proposition.

93. Prop. *To determine the conditions of equilibrium of forces any how situated applied to any lever.*

We have here a system of forces with a fixed point, whence, as in art. 49, the forces must be reducible to a single resultant passing through the fulcrum; and they must have no resulting moment relative to that point.

The pressure on the fulcrum = resultant of the forces.

94. *Cor.* If the fulcrum is a cylindrical axle, and the lever can only move in one plane, then, as in art. 50, the only condition is, that the moments of all the forces reduced to that plane should equal zero.

The pressure on the axis is in this case determined from the equations X and Y (32).

95. In the ordinary applications of the lever the *forces are situated in the same vertical plane, and the lever is only capable of moving in that plane.* In this case it is easy to determine the conditions of equilibrium.

Let fall perpendiculars from the fulcrum on the directions of the forces, then if

p', p'', p''', &c. represent the forces.
q', q'', q''', &c. represent the perpendiculars,

we have (15)

$$p'q' + p''q'' + p'''q''' + \&c. = 0,$$

or the sums of the moments of all the forces round the fulcrum must = 0. Those which tend to turn the lever in contrary directions have of course different signs.

96. This condition may be more conveniently expressed, thus,

let r', r'', r''', &c. be the distance of the points of application of the forces from the fulcrum.
α', α'', α''', &c. the angles made by the directions of the forces with the distances.

Then since $q' = r' \sin. \alpha'$, &c.

$$p'r' \sin. \alpha' + p''r'' \sin. \alpha'' + p'''r''' \sin \alpha''' + \&c. = 0,$$
$$\text{or } \Sigma.\, p.r. \sin. \alpha = 0$$

is the equation of equilibrium.

In this formula, the distances on the one side of the fulcrum must be taken as positive, and those on the other as negative. Also those forces which act *upwards* must have different signs from those which act *downwards*.

The pressure on the fulcrum is equal to the resultant of all the forces.

97. *Cor.* 1. If the forces act in parallel directions,

$$\alpha' = \alpha'' = \alpha''' \ \&c.$$

and we have

$$p'r' + p''r'' + p'''r''' + \&c. = 0.$$

98. *Cor.* 2. If there are only two forces parallel to each other

$$\therefore\ p'r' + p''r'' = 0.$$

Hence either one of the forces must be *negative* or the points of application on contrary sides of the fulcrum, so that

$$p'r' = p''r''$$
$$\text{or } p' : p'' :: r'' : r'$$

or the forces are reciprocally proportional to their distances from the fulcrum. Compare art. 21.

99. In practical applications where only a *power* and *weight* (90) are introduced, it is usual to divide levers into *three* kinds, according to the relative positions of the power, weight, and fulcrum.

In a lever of the *first* kind, the *fulcrum* is between the *power* and the *weight.*

In the *second,* the *weight* is between the *power* and the *fulcrum.*

In the *third,* the *power* is between the *weight* and the *fulcrum.*

100. We sometimes have a combination of levers presented to us, either all of one kind, or different, as circumstances require. The conditions of equilibrium of this compound machine are easily investigated. For the effect or weight of the first, is as it were the power upon the second. Thus if ABD (fig. 19.) represent a system of two levers, F and F′ being the fulcra, we have (98)

$$\text{P at A} : \text{P at B} :: \text{FB} : \text{FA}.$$
$$\text{P at B} : \text{P at D} :: \text{F}'\text{D} : \text{F}'\text{B}.$$
$$\therefore\ \text{P at A} : \text{P at D} :: \text{FB}.\,\text{F}'\text{D} : \text{FA}.\,\text{F}'\text{B}.$$

and similarly for any number of levers.

CHAPTER II. §. II.

THE BALANCE.

101. The *balance* is a lever, and generally of the first kind. Its object is to ascertain the weight of any body.

In its usual construction the balance has two equal arms, at the extremities of which pans or scales are suspended, into one of which the body whose weight is required is placed, and into the other bodies whose weights are known.

Since the arms are equal, an equilibrium will exist if equal weights are placed at each end (98); but *rest* alone is not sufficient. For if the fulcrum were placed at, or if it were suspended by the centre of gravity, it would rest in any position.

The requisites of a good balance are,

1st. The *horizontality* of the beam or bar.

2ndly. The *sensibility* of the system.

3rdly. The *stability* of the system.

102. The beam will remain at rest in a *horizontal* position, if the centre of gravity of the system be below the point of suspension in the perpendicular to the beam at that point (75), which it will be if the arms and scales are equal and of equal weight.

103. The *sensibility* of the balance is measured by the angle, which the addition of a very small weight in either scale would cause the arm to describe from its horizontal position.

To estimate this, let A B (fig. 20) represent the arms of the balance, S the point of suspension, G the centre of gravity of the system when a weight equal to P is in each scale: let also W = the weight of the system when unloaded. By the addition of a small weight, w, in the scale at A, the extremity A will be depressed, and the centre of gravity of the system will take a different position, as G′ (fig. 21) in the vertical through S. Then if HSK be drawn parallel to the horizon through S, the ∠ GSG′ which = ∠ ASH measures the deviation of the beam from its horizontal position.

The point G′ is now the centre of gravity of 2 P + W which may be considered as placed at G, and of w placed at A.

Hence (96) G K being a vertical line,

$$w. \text{SA. sin. SAH} - (2\text{ P} + \text{W}). \text{SG. sin. SGK} = 0$$

or if SA $= r$, SG $= b$, and ∠ ASH $= \theta$.

$$\therefore \text{sin. SAH} = \text{cos. } \theta. \text{ and sin. SGK} = \text{sin. } \theta.$$

Therefore we have,

$$\text{tan. } \theta. = \frac{r.\ w}{(2\text{ P} + \text{W}).\ b.}$$

The sensibility is therefore increased, either (1) by increasing the length of the arms, or (2) by diminishing the distance between the point of suspension and the centre of gravity of the system, or (3) by diminishing the weight of the beam or load.

104. The *stability* is measured by the tendency to return to its position of equilibrium, if it is disturbed.

If the balance is brought into the position represented in fig. 21., we have the weights 2 P + W applied at G tending to restore the horizontality of the bar; and the *moment* of these weights is (2 P + W) b. sin. θ.

Hence by increasing the sensibility we diminish the stability. The only way of combining the requisites is to lengthen the arms without increasing their weight, which may be at the expense of their rigidity or strength.

In the preceding investigations we have supposed the line A B to pass through S the point of suspension. In practice this is not the case, but the error arising from the supposition is inconsiderable.

105. The *Steelyard* is a balance with one arm longer than the other. A given weight is moveable upon the longer arm, and the body to be weighed is attached to the extremity of the shorter (fig. 22).

106. The *Danish Steelyard* has a given weight attached to one end, and the body to be weighed is attached to the other, the fulcrum being moved to establish equilibrium.

But enough has been said to shew the principles of these and other contrivances.

CHAPTER II. §. III.

THE WHEEL AND AXLE.

107. The *Wheel and Axle* consists of a cylinder or axle AB, (fig. 23.) and a circle or wheel, whose centre is on the axis of the cylinder. The whole is supposed to be perfectly rigid and to be moveable only round the axis of the cylinder.

108. Prop. *To determine the conditions of equilibrium of any forces acting on the wheel and axle.*

We have here a system of forces with a fixed axis, whence as in art. 50. the requisite condition is that there should be no moment round the axis.

109. But in the ordinary applications of the wheel and axle, the power is applied to the surface of the wheel, and the weight to the surface of the cylinder. To ascertain the conditions of equilibrium in this case, we may suppose the power and weight to act in the same plane, since the whole system is assumed to be perfectly rigid, and therefore any motion communicated to one part of the surface of the cylinder is communicated to every part. We have then a simple case of the lever, and therefore if P and W are the power and weight (98.)

$$\begin{aligned} P : W &:: CA : CB \text{ (fig. 24.)} \\ &:: \text{rad. of axle} : \text{rad. of wheel.} \end{aligned}$$

110. If ropes are used for the action of the power and weight, we must suppose the forces applied to the axis of the ropes. In

this case if t, and t' represent half the thickness of the ropes, and r and r' the radii of the axle and wheel

$$P : W :: r + t : r' + t'.$$

111. If the power and weight are on opposite sides of the axis, the pressure on the axis equals the sum of the two; if they are on the same side the pressure equals their difference.

CHAPTER II. §. IV.

THE CORD.

112. The *Cord* or *Rope* is useful as a means of communication of force; and if we introduce merely a simple cord no other advantage is gained, since the forces at each end must be equal and opposite in the case of equilibrium.

But by passing the cord over, or attaching it to certain fixed points, the resistance of these points thus excited may be employed advantageously.

113. Prop. *To determine the conditions of equilibrium of a cord acted on by any forces applied at different points of the cord.*

Let A B C D E (fig. 25.) be a cord of any length drawn by different forces p', p'', p''', &c. applied at the points A, B, C, D, &c. and acting in the directions Ap', Bp'', Cp''', &c.

Now equilibrium cannot exist unless each of the parts of the cord BC, CD, &c. are drawn at each extremity by equal and opposite forces in the directions of the parts. Hence the resultant of p' and p'' must coincide with C B produced, and this resulting force in the direction B K may be considered as applied at any point in its direction, and therefore may be supposed to act at C. When it is transferred to that point it may be again resolved into two, in directions respectively parallel to the original components, as Cm', Cm''. Similarly the resultant of the three p', p'', p''', may be transferred to the point D, and again decomposed; and so on for any number.

Hence the condition of equilibrium is, *that the forces if transferred to the same point, and supposed to act at that point in* *directions parallel to their own directions, should keep the point in equilibrio.*

Therefore if α', β', γ' be the angles which p' makes with three axes.

&c. &c.

the equations of art. 18 apply here, viz.

$$\Sigma.\ p.\ \cos.\ \alpha = 0. \qquad \Sigma.\ p.\ \cos.\ \beta. = 0. \qquad \Sigma.\ p.\ \cos\ \gamma. = 0.$$

114. But it may be more convenient to consider distinctly the case of a cord acted on by three forces.

Let A C B (fig. 26.) be a cord drawn by two forces p', p''', at its extremities, and another force p'' at any intermediate point C, in the direction Cp''. Complete the parallelogram C M K N having its diagonal K C in the direction of the force p', and the other sides in the direction of the other forces. Hence if $\angle$ A C K $= \alpha$, and $\angle$ K C B $= \alpha'$ we have in the case of equilibrium (10)

$$\begin{aligned} p' : p'' : p''' &:: \mathrm{CM} : \mathrm{CK} : \mathrm{CN} \\ &:: \sin.\ \mathrm{KCN} : \sin.\ \mathrm{MCN} : \sin.\ \mathrm{MCK} \\ &:: \sin.\ \alpha' : \sin.\ (\alpha + \alpha') : \sin.\ \alpha. \end{aligned}$$

115. *Cor.* 1. If the cord be fixed at the points A and B, the reactions of the points take the place of the forces. These reactions are also equal to the tensions of the two parts of the cord.

116. *Cor.* 2. If the force p'' which is applied at C, be applied to a running knot or ring, the points A and B being fixed, the ring will not remain at rest until the direction of the force bisects the angle made by the two parts of the cord.

For the ring in its descent will describe an ellipse (fig. 27.), A and B being the foci, and the force will not be destroyed, or in equilibrio, unless it be a normal to the curve: or unless

$$\angle \text{ACK} = \angle \text{BCK}.$$

$$\therefore \text{ pressure on A} = \text{pressure on B}.$$

and since $p' : p'' :: \text{sin. } \alpha' : \text{sin. } (\alpha + \alpha')$. (114.)

$$:: \text{sin. } \alpha : \text{sin. } 2\,\alpha$$

$$:: 1 : 2 \text{ cos. } \alpha$$

$\therefore p'' = 2\,p' \text{ cos. } \alpha$. p' denoting the pressure at A or B.

The same would apply if the cord instead of being fixed at A and B passed over those points, and had weights attached to each end (fig. 28). The demonstration shews that in the last case, the weights must be equal.

But if the cord were bent at a sharp angle, a great loss of force would be occasioned by the rigidity and friction of the cord; to obviate which, the pulley has been contrived.

CHAPTER II. §. V.

THE PULLEY.

117. The *Pulley* is a small grooved wheel moveable about an axis and fixed in a *block*. The cord passes over the circumference of the wheel in the groove.

The use of the pulley is to prevent the effects of friction, and the rigidity of the cord. The first of these it diminishes by transferring the friction from the cord and the circumference of the wheel to the axle and its supports, which may be highly polished. The effects of the rigidity are diminished by turning the cord in a circular arc instead of at a sharp angle.

118. The conditions of equilibrium of a cord passing over a pulley, are the same as those of a cord passing over a fixed point, or through a ring.

In the single *fixed* pulley (fig. 29.) no advantage is gained, except convenience in the application of force, and here

$$P = W.$$

In the single *moveable* pulley (fig. 30.) we have by art. 116,

$$P = \frac{W}{2 \cos. \alpha.} \text{ generally.}$$

But if the ropes are parallel $\alpha = 0$.

$$\therefore P = \frac{W}{2}.$$

This supposes the weight of the block itself to be inconsiderable, but if it cannot be neglected, it must be added to the weight to be sustained, and is therefore a disadvantage.

119. By combinations of fixed and moveable pulleys, the power is rendered capable of sustaining a greater weight, and may be applied to greater advantage, and the ratio of the *power* to the *weight* in some of the more useful combinations may be easily deduced.

In the system (fig. 31.) where one is fixed and one moveable, no other advantage is gained by the introduction of the fixed one, than convenience in the application of the power, which may then be another weight. And if w = the weight of the moveable block, we have (118.), if the parts of the ropes are parallel.

$$2\,P = W + w.$$

120. In the system (fig. 32.) where there are two blocks of pulleys, the upper one fixed, and the lower one moveable, where the same rope passes round all the pulleys, and the weight is attached to the lower block, and the power is applied to the extremity of the rope, we have merely a combination of the last case, and therefore

$$n\,P = W + w$$

where n is the number of ropes at the lower block and w = weight of the lower block.

If the ropes are not parallel, we must introduce the cosine of the angle made with the vertical in each case (118), but the angle is generally so small that it may be neglected.

121. In the system (fig. 33.) where each pulley hangs by a separate rope, we have merely a combination of the single moveable in art. 118.

Hence if the ropes are parallel

$$2^n\,P = W + w' + 2\,w'' + 2^2\,w''' + \&c.$$

where n is number of moveable pulleys, and w', w'', w''', &c. the weights of each block.

If the ropes are not parallel we must proceed as in the last case.

122. In the system (fig. 34.) where each rope is attached to the weight.

If the ropes are parallel we have

$$W = (2^n - 1) P + (2^{n-1} - 1) w' + (2^{n-2} - 1) w'' + \&c.$$

Where n is the number of pulleys and w', w'' w''', &c. the weights of each.

In this system the weights of the pulleys are on the side of the *power*.

123. With respect to the pressures on the fixed points we at once perceive that in the single fixed pulley the pressure on the axis $= 2 P \cos. a.$ (116) if a be half the angle of inclination of the ropes.

In the system in art. 119, the fixed points sustain $P + W + w$, and the pressure on B is twice the pressure on A.

These principles will be easily applied to the other systems.

CHAPTER II. §. VI.

THE INCLINED PLANE.

124. The *inclined plane* assists by its reaction in sustaining a heavy body.

A body on inclined plane will, if left to itself, descend (89); since the reaction of the surface which is in the direction of the normal (48) cannot entirely counteract the weight of the body. Some other force must therefore be made to act upon the body, in order that it may be sustained.

125. *Prop. To determine the conditions of equilibrium of a body sustained by any force on an inclined plane.*

Let K L M (fig. 35.) be a body on an inclined plane A C, then its weight W may be considered as acting at G the centre of gravity of the body in a vertical direction. If then a force P acts upon it, the first condition is that the forces P and W should have a single resultant; hence their directions must meet in some point. And since W acts in the vertical direction, the first condition will be fulfilled, *if the force P acts in the vertical plane passing through the centre of gravity of the body.*

The second condition is, that the resultant of the forces should be a normal to the plane, in order that it may be counteracted by the reaction; hence *the vertical plane through the centre of gravity must pass through the surface of contact of the body, and be perpendicular to the plane,* i. e. *perpendicular to the intersection of the inclined and horizontal planes.*

If therefore G P, G H, and G N (fig. 36.), represent the directions of the three forces, viz. the acting force (P), the weight of the body (W), and the reaction of the plane (N), we have by the parallelogram (9),

$$\begin{aligned} \mathrm{P} : \mathrm{W} : \mathrm{N} &:: \mathrm{GI} : \mathrm{GF} : \mathrm{GE} \\ &:: \text{sin. } \mathrm{IEG} : \text{sin. } \mathrm{GEF} : \text{sin. } \mathrm{GFE} \\ &:: \text{sin. } \mathrm{BAC} : \text{sin. } \mathrm{PGE} : \text{sin. } \mathrm{PGH}. \end{aligned}$$

If several forces act on the body, P must be considered as their resultant.

126. *Cor.* 1. Hence P is capable of sustaining the greatest weight, when sin. P G E is the greatest, or when P is parallel to the plane. In this case we have

$$\begin{aligned} \mathrm{P} : \mathrm{W} : \mathrm{N} &:: \mathrm{CB} : \mathrm{CA} : \mathrm{AB} \\ &:: \text{height} : \text{length} : \text{base}. \end{aligned}$$

127. *Cor.* 3. If P act parallel to the horizon,

$$\begin{aligned} \mathrm{P} : \mathrm{W} : \mathrm{N} &:: \mathrm{CB} : \mathrm{AB} : \mathrm{CA} \\ &:: \text{height} : \text{base} : \text{length}. \end{aligned}$$

128. If two inclined planes of equal height are placed as it were back to back, and weights W, W′ support each other on them by means of a cord over a pulley or wheel on the top, so that the parts of the cord are parallel to the planes, as in fig. 37, Then if L and L′ are their lengths, and H their common height, the power necessary to sustain each weight (126) $\mathrm{W}.\frac{\mathrm{H}}{\mathrm{L}}$

$$\therefore\ \mathrm{W}.\frac{\mathrm{H}}{\mathrm{L}} = \mathrm{W}'.\frac{\mathrm{H}}{\mathrm{L}'}$$

$$\therefore\ \mathrm{W} : \mathrm{W}' :: \mathrm{L} : \mathrm{L}'.$$

129. If a body be sustained between two inclined planes which meet at an angle (fig. 38.), its weight is counteracted by their reactions; hence the directions of the three forces must meet in a point, and the conditions of equilibrium may be investigated as in the above cases.

CHAPTER II. §. VII.

THE WEDGE.

130. The *Wedge* is a triangular prism and is generally used to separate obstacles by introducing one of its edges (called the *angle* of the wedge) between them.

The power is exerted on the back or surface opposite to the *angle;* and we may suppose its direction to be perpendicular to the angle. The weight is in this case the resistances opposed by the obstacles to the two *faces* of the wedge which contain the angle: and if we suppose the faces to be perfectly smooth, the only effective parts of the resistances must be perpendicular to the surfaces.

131. Prop. *To determine the conditions of equilibrium of the wedge.*

Let ABC (fig. 39.) be a section of the wedge, and let a power P act on the back AB in the direction PDE, let MD and ND be perpendicular to the sides and therefore in the direction of the resistances, then taking any line DE in the proper direction to represent the power and completing the parallelogram DIEK, the sides DI, DK represent the resistances.

∴ If W, W′ represent the resistances

$$\begin{aligned} P : W : W' &:: DE : DI : IE \\ &:: AB : AC : BC, \text{ by sim. triangles.} \end{aligned}$$

$$\therefore P : W + W' :: AB : AC + BC.$$

132. *Cor.* If the wedge is isosceles, or the section an isosceles triangle

$$\therefore P : W + W' :: AB : 2AC$$
$$:: \sin. \tfrac{1}{2} ACB : \text{radius}.$$

In this case $W = W'$.

133. In investigating the theory of the equilibrium of the wedge, we have omitted the consideration of the *friction*, and we have also supposed the power to be a *pressure*. Whereas in practice the wedge is kept at rest by *friction* alone, and the power arises from *percussion*.

CHAPTER II. §. VIII.

THE SCREW.

134. If we divide the rectangle A B C D (fig. 40.) into equal parts by the lines $m n$, $m' n'$, &c. parallel to the sides, and draw the parallel diagonals of the rectangles thus formed, and if we suppose this rectangle to be wrapped round the surface of a cylinder, the perimeter of whose base is equal to A B, and altitude to B C, the diagonals of the rectangles will trace on the surface of the cylinder a *continuous* curve, which is called the *helix*[a].

If a projecting *thread* or *rib* be attached to the cylinder upon this curve, we have the *external screw* or as it is more properly called the *screw*. Similarly if we take a hollow cylinder of exactly the same radius as the solid one, and generate a groove in the same curve, we have the *internal screw* or *nut*.

The screw is placed in the nut, which is generally fixed, though the circumstances of the problem will be the same whichever be moveable.

135. *Prop. To determine the conditions of equilibrium on the screw.*

From the construction it appears that the thread of the screw is an inclined plane. The force is generally applied by means of a lever attached to the top of the screw, and since the whole system may be supposed perfectly rigid, we may in the investigation suppose the force to act in the same horizontal plane as that in which the resistance against the thread of the screw is situated.

---

[a] For a discussion of this curve, see Cauchy, Applicat. du Calcul. Infinitesimal, xiii. or Hymer's Analytical Geometry, 34.

If then E D B (fig. 41.) is a horizontal section of the screw through B the point where the resistance or weight is supported, and if C B A represent the position of the lever, we have a weight sustained at B on an inclined plane by a force parallel to the base.

If therefore r = radius of cylinder.
R = radius of circle described by the power P.
h = distance between two threads = $m\,m'$.

By art. 127.

$$\begin{aligned} P \text{ at } B : W &:: \text{height of plane} : \text{base} \\ &:: h : 2\pi r \end{aligned}$$

and by art. 98

$$\begin{aligned} P : P \text{ at } B &:: r : R. \\ \therefore\ P : W &:: h : 2\pi R. \end{aligned}$$

We have here supposed the whole resistance sustained at one point in the thread, but if the resistance be distributed over one or more threads, we have

$$\begin{aligned} \text{Each part of } P : \text{the part of } W \text{ sustained by it} &:: h : 2\pi R \\ \therefore\ P : W &:: h : 2\pi R. \end{aligned}$$

CHAPTER III.

ON THE FUNICULAR POLYGON, AND CATENARY.

136. The investigations in the last chapter, on cords, if extended, lead to some remarkable results.

The polygon formed by forces acting at different points of a cord, is called *the funicular polygon.*

To the condition of equilibrium before deduced (113) it must be added, that, as the cord is *inextensible,* or incapable of being lengthened, but not *incompressible* or incapable of bending, the forces must be *divellent,* i. e. they must act *from within towards without.*

But if the sides of the polygon are so many *incompressible* bars simply resting against each other, without joints at the angles, it will be necessary that the forces act *from without towards within,* urging the angles to approach each other.

But if they are rigid bars with joints at the angles, so that the angles may open or close, but the extremities of the sides cannot separate, it will be indifferent in which direction the forces act.

137. *Prop. To compare the forces which acting on a cord keep it in equilibrio.*

Let p''B (fig. 42.) be produced to N,

and let the $\angle\ p'\mathrm{BN} = \alpha$ and so the $\angle\ \mathrm{BCM} = \beta$

$\angle\ \mathrm{NBC} = \alpha'$ $\angle\ \mathrm{MCD} = \beta'$, &c.

Let t be the tension of C B or the resultant of p' and p''.

$$\therefore (114)\ p' : p'' : t :: \sin.\ \alpha' : \sin.\ (\alpha + \alpha') : \sin.\ \alpha$$

$$\text{also}\ t : p''' : t' :: \sin.\ \beta' : \sin.\ (\beta + \beta') : \sin.\ \beta.$$

&c. &c.

From the first of these proportions we obtain

$$p' = \frac{p'' \sin. \alpha'}{\sin. (\alpha + \alpha')} \qquad t = \frac{p'' \sin. \alpha}{\sin. (\alpha + \alpha')}$$

and since t in the first proportion is the same as t in the second, &c.

$$\frac{p'' \sin. \alpha}{\sin. (\alpha + \alpha')} = \frac{p''' \sin. \beta'}{\sin. (\beta + \beta')}$$

$$\frac{p''' \sin. \beta}{\sin. (\beta + \beta')} = \frac{p'''' \sin. \gamma'}{\sin. (\gamma + \gamma')}$$

If the cord be fixed at the points A and E, the value of p' is the tension of the part B A.

138. *Cor.* 1. If when the cord is suspended at A and E the directions of the forces are all parallel, we have

$$\sin. \alpha' = \sin. \beta, \qquad \sin. \beta' = \sin. \gamma, \text{ \&c.}$$

Multiplying therefore these by the expressions for the tensions of the strings in the last article, we have

$$\frac{p'' \sin. \alpha \sin. \alpha'}{\sin. (\alpha + \alpha')} = \frac{p''' \sin. \beta \sin. \beta'}{\sin. (\beta + \beta')}$$

$$= \frac{p'''' \sin. \gamma \sin. \gamma'}{\sin. (\gamma + \gamma')} \text{ \&c.}$$

$$\text{or } \frac{p''}{\cot. \alpha + \cot. \alpha'} = \frac{p'''}{\cot. \beta + \cot. \beta'} = \frac{p''''}{\cot. \gamma + \cot. \gamma'} \text{ \&c.}$$

In this case the directions of the forces and the whole polygon necessarily lie in the same plane.

139. *Cor* 2. If the forces p'', p''', &c. are weights suspended at the points B, C, D, &c. the whole will necessarily lie in the vertical plane passing through the points of suspension.

If in this case the tension of any one of the strings is resolved into two, one vertical and the other horizontal, as for instance B A (fig. 43.) into B N, N A;

the horizontal tension N A = A B. sin. A B N = p' sin. α

so the horizontal tension of C B = t sin. β.

Substituting for p' and t their values found in art. 137, we find the horizontal tension of B A $= \frac{p''}{\text{cot. } \alpha + \text{cot. } \alpha'}$

the horizontal tension of C B $= \frac{p'''}{\text{cot. } \beta + \text{cot. } \beta'}$ &c.

and therefore (138) they are all equal to one another.

Hence the horizontal tension is constant throughout the whole polygon.

140. *Cor.* 3. Since the reactions of the points A and E equilibrate the resultant of all the weights, the lines A B, E D produced must meet one another in some point of the vertical through the centre of gravity of the system. For three forces, if in equilibrio, must meet in a point.

141. A heavy cord may be considered a funicular polygon loaded with an infinite number of small weights, and since the number of weights is infinite, the polygon will also have an infinite number of sides, or will become a curve.

This curve, which a heavy cord or chain of indefinitely small links will assume when suspended from two fixed points not in the same vertical line, is called the *catenary*.

142. *Prop.* *To find the equation to the catenary.*

Le K and K′ (fig. 44.) be the points of suspension, and taking the lowest point A of the curve as origin, and the vertical and horizontal lines passing through it as axes, let $AM = x, \qquad PM = y, \qquad AP = s,$

Now the equilibrium will subsist, if the whole arc A K′ be supposed to be removed, provided that a horizontal tension is applied at A equal to that which would be produced by the arc; which horizontal tension is constant throughout the whole system (139)

If then P T is the tangent at P, the three sides of the triangle P T M represent respectively the tension of P K, the weight of A P, and the horizontal tension of A K′.

$$\therefore \text{ weight of } AP\ (= W) = \frac{MT}{MP} \cdot \text{ horizontal tension of } AK'$$

$$= \frac{dx}{dy} \cdot ck$$

where c denotes a length of the curve, the weight of which would be equal to the horizontal tension, and k is the weight in the unit of length.

If the curve is uniform in density, or the catenary is *homogeneous,*

$$W = ks.$$

$$\text{Hence } s = c\,\frac{dx}{dy} \quad \therefore \quad \frac{c}{s} = \frac{dy}{dx}$$

which is the differential equation of the catenary.

To integrate this, we obtain by squaring both sides and adding unity to each,

$$\frac{c^2 + s^2}{s^2} = \frac{dy^2 + dx^2}{dx^2} = \frac{ds^2}{dx^2}$$

$$\therefore dx = \frac{s\,ds}{\sqrt{c^2 + s^2}}.$$

Hence, since if $x = 0 \therefore s = 0$

$$\therefore x + c = \sqrt{c^2 + s^2}$$

$$\therefore s = \sqrt{x^2 + 2cx}.$$

143. *Prop. To find the equation between y and s.*

By inverting the differential equation, squaring both sides, and adding unity to each,

$$\frac{s^2 + c^2}{c^2} = \frac{dx^2 + dy^2}{dy^2} = \frac{ds^2}{dy^2}$$

$$\therefore dy = \frac{c\,ds}{\sqrt{c^2 + s^2}}$$

Hence integrating and correcting as above,

$$y = c.\ h.l.\ \frac{s + \sqrt{c^2 + s^2}}{c}.$$

144. *Prop. To find the equation between x and y.*

Substitute in the last equation the value of s found in the preceding case,

$$\therefore y = c.\ h.l.\ \frac{x + c + \sqrt{x^2 + 2cx}}{c}.$$

DYNAMICS.

INTRODUCTION.

145. *Motion* or the passage from place to place (2) must be accompanied with a determinate *rate.*

The *duration* of the passage may be greater or less, and the *greater* the duration or *time,* the slower the motion, and the *less* the *rate.* The distance of the two places may also vary, although the duration of the motion remain the same, in which case the *greater* the distance or *space,* the *greater* the *rate.*

146. This rate of motion is called the *velocity,* and although velocity is not itself a mathematical quantity, yet as our ideas of it are formed from the duration or time of the motion, and the space passed over, it is at once brought within the province of mathematical reasoning.

Velocity then depends on the time and the space, and by fixing either on some given time, or some given space, a comparison may be made between two or more velocities.

It is more usual and convenient to assume some unit of time for the comparison, and in mathematical reasonings one second is fixed upon.

Velocity then is measured by the space described in the unit of time.

In any other time t we obtain the equation

$$v = \frac{s}{t}$$

This supposes the motion to be *uniform* or equal spaces to be described in equal successive portions of time.

If the spaces thus successively described are not equal, the motion is *variable,* and either *accelerated* or *retarded,* according as the spaces form an *increasing* or *decreasing* series.

In the case of variable motion, the velocity is naturally measured at each instant by the limiting ratio of the increment of the space to the time in which that increment is described.

Hence by the nature of the differential coefficient,

$$v = \frac{ds}{dt} \qquad \text{(I.)}$$

Velocity therefore generally is *measured by the space that would be uniformly described in a given time,* were all disturbing causes at that instant removed.

147. The cause of motion has been called (3) *force,* which force is considered as applied to a point or elementary particle of matter. These elementary particles we assume to be perfectly *passive,* i. e. to have no force or power resident in them by which they can impede or any way modify the effect of any force or impressed motion[a].

Hence if a force act on a point during an indefinitely small time, and, having communicated a determinate velocity, cease to act upon it, the effect of such a force is *equable* and *uniform* motion in the direction of the force.

---

[a] For the deduction of the equations between the force, time, and velocity &c. and for the establishment of the theory of motion, two assumptions hav been made, equivalent to the two first laws of motion, viz. the *inertia* of th point or elementary particle of matter, and secondly, that the effect of a force proportional to the force, and in the direction of the force. That such are t laws according to which matter actually moves will be considered in the prop place.

If a force *continually* act upon a point, the velocity is at every instant increased and the effect is *variable* motion in the direction of the force. Such a force is an *accelerating* force, and if the alteration of velocity is *constant,* i. e. if equal velocities are added in equal successive times, the force is called an *uniformly accelerating force,* or simply an *uniform force.*

Such a force is therefore *measured by the velocity generated in any time,* and here also it is most convenient to assume one second, as the unit of time for a standard of comparison.

Hence an *uniform force, is measured by the velocity generated in the unit of time.*

In any other time t we obtain the equation

$$f = \frac{v}{t}$$

If the force is not uniform, we have by the differential coefficient,

$$f = \frac{dv}{dt} \qquad \text{(II.)}$$

148. By differentiating equation (I.) we obtain $\dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$

$$\therefore f = \frac{d^2s}{dt^2} \qquad \text{(III.)}$$

149. By eliminating dt between (I.) and (II.) we obtain

$$vdv = fds. \qquad \text{(IV.)}$$

DYNAMICS.

SECTION I.

CHAP. I.

ON THE MOTION OF A PERFECTLY FREE POINT.

150. If a single force, or several forces in the same or exactly opposite directions act upon a free point, motion can only ensue in the line of the direction of the forces. But if several forces in different directions *continually* act upon it, the point will have a resulting motion, whose direction and rate must be determined by the relation between the directions and intensities of the forces which act upon it. The motion may or may not be rectilinear, and in order to investigate the several circumstances of the resulting motion generally, we introduce the method of resolution to three rectangular axes, as in Statics.

151. Prop. *To determine the motion of a point acted on by forces in any directions.*

Resolve the forces or motions according to three rectangular axes, and let X, Y, Z, represent the sums of the resolved forces in each axis, the position of the point being determined by the three coordinates x, y, z. The three forces X, Y, Z, being at right angles to each other, are independent of one another, and may therefore be considered as acting on the point separately: hence by equation (II.)

$$X = \frac{d^2 x}{d t^2}, \qquad Y = \frac{d^2 y}{d t^2}, \qquad Z = \frac{d^2 z}{d t^2}, \qquad \text{(V.)}$$

Applying the same principles to the velocity of the point, we have, if $d s$ represent any small element of the space or path of the body, the velocity in the path $= \frac{d s}{d t}$ by equation (I.)

Hence resolving this similarly, the velocities in each axis are

$$\frac{d x}{d t}, \qquad \frac{d y}{d t}, \qquad \frac{d z}{d t}.$$

Such are the general equations by which the motion of a point in space may be determined.

If the velocity is given, we must by resolution obtain the velocity in each axis, and thence by differentiation the force in each axis, and by composition the resultant force may be determined.

If the forces are given, we must obtain the velocity in each axis, and hence the velocity in space.

If the path is required, each of the equations (V.) must undergo two integrations, at each of which operations a constant must be introduced. The constants introduced at the first integration will depend on the *velocity* of the point; those at the second on the *position* of the point. We thus obtain three equations, involving the coordinates x, y, z, and t, and by the elimination of t, two equations between the coordinates will be obtained, which are the equations to the *path* of the point. This path is *generally* a curve of double curvature, but in particular cases the investigations lead to a more simple result.

152. *Prop. To determine the velocity of the point in its path.*

We have stated in the last proposition how the velocity may be found, but there is another more elegant method of determining it, thus:

Multiply the first of equations (V.) by $2\,dx$, the second by $2\,dy$, the third by $2\,dz$, and adding the results together we have

$$\therefore \frac{2\,dx\,d^2x + 2\,dy\,d^2y + 2\,dz\,d^2z}{dt^2} = 2(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz)$$

The first member of this equation is the exact differential of

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} \text{ which } = \frac{ds^2}{dt^2} = v^2. \text{ by (I.)}$$

Therefore integrating we find

$$v^2 = 2\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) + \mathrm{C}. \quad \text{(VI.)}$$

153. *Cor.* If the expression $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ is integrable, the velocity at any point may be found, if we are able to correct the integral, or if we know the velocity at some given point.

Thus, if the expression is a differential of the coordinates,

$$v^2 = \mathrm{C} + 2f(x, y, z),$$

and if v' be the velocity at any point whose coordinates are a, b, c,

$$\therefore v'^2 = \mathrm{C} + 2f(a, b, c),$$

$$\therefore v^2 - v'^2 = 2f(x, y, z) - 2f(a, b, c).$$

Hence it appears that the velocity acquired by the particle in passing from one point $(a, b, c,)$ to any other, is the same whatever be the curve described between those points.

154. The deduction in the last article supposes that the expression $\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz$ is an exact differential; and it is remarkable that *whenever the forces are directed to a fixed centre, and when the intensity of the force is a function of the distance from that centre, the expression is always integrable.*

For, let F represent the intensity of such a force directed to a fixed centre, let $a, b, c,$ be the coordinates of the centre,

$x, y, z,$	the coordinates of any position of the particle,
r	the distance of the particle from the centre,
$\alpha, \beta, \gamma,$	the angles which r makes with the three axes.

Then resolving F into directions parallel to each axis, we have for the three components

$$F. \cos. \alpha, \qquad F. \cos. \beta, \qquad F. \cos. \gamma.$$

But $\cos. \alpha = \frac{x-a}{r}, \qquad \cos. \beta = \frac{y-b}{r}, \qquad \cos. \gamma = \frac{z-c}{r}.$

Hence the force F is equivalent to

$$F. \frac{x-a}{r}, \qquad F. \frac{y-b}{r}, \qquad F. \frac{z-c}{r}.$$

Similarly any other forces F′, F″, &c. may be resolved.

Therefore the expression $Xdx + Ydy + Zdz =$

$$F\left(\frac{x-a}{r}, dx + \frac{y-b}{r}. dy + \frac{z-c}{r}. dz\right) +$$

$$F'\left(\frac{x-a}{r}. dx + \frac{y-b}{r}. dy + \frac{z-c}{r}. dz\right) + \&c.$$

and since $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$

we have by differentiation

$$dr = \frac{x-a}{r} dx + \frac{y-b}{r} dy + \frac{z-c}{r} dz$$

$$\therefore Xdx + Ydy + Zdz = Fdr + F'dr + F''dr, \&c.$$

and since by hypothesis each force F is a function of the distance

r, each of the terms of the second member of the equation is an exact differential, and therefore the first is also.

155. *Cor.* If any of the forces were directed to a *moveable centre*, the coordinates, a, b, c, would not be constant, the value of dr would not be complete, and therefore the formula would not be an exact differential.

156. Prop. *To determine the motion of a point acted on by a force tending to a fixed centre.*

Here it will be most convenient to assume the fixed point as the origin.

Let F denote the force, whether a single force or the resultant of several.

Let r denote the distance of any position of the particle from the centre.

Then if we multiply the first of equations (V.) by y, and the second by x, we obtain by subtraction

$$\frac{xd^2y - yd^2x}{dt^2} = x\mathrm{Y} - y\mathrm{X}.$$

Proceeding similarly for the other combinations we obtain

$$\frac{zd^2x - xd^2z}{dt^2} = z\mathrm{X} - x\mathrm{Z},$$

$$\frac{yd^2z - zd^2y}{dt^2} = y\mathrm{Z} - z\mathrm{Y}.$$

But by resolving F according to the three axes, the components are

$$\mathrm{X} = \mathrm{F}.\frac{x}{r}, \qquad \mathrm{Y} = \mathrm{F}.\frac{y}{r}, \qquad \mathrm{Z} = \mathrm{F}.\frac{z}{r}.$$

From which we easily find

$$yX = xY, \qquad zX = xZ, \qquad yZ = zY.$$

The second members therefore of the first equations are each $= 0$, and by integrating the first members which are exact differentials, we find

$$\begin{aligned} xdy - ydx &= cdt, \\ zdx - xdz &= c'dt, \\ ydz - zdy &= c''dt. \end{aligned}$$

Again multiplying the first of these equations by z, the second by y, and the third by x, and adding

$$cz + c'y + c''x = 0$$

which is the equation to a *plane*.

Hence the *path* or *trajectory* is a *plane curve* passing through origin of coordinates or centre of force.

157. *Cor.* 1. With respect to the results of the integration in the last article, we may observe, that if

r = radius vector or distance of the projection of the point on the plane of x, y, from the centre or fixed pole,

ω = the angle made by r with the axis of x

$$\therefore\ xdy - ydx = r^2 d\omega = cdt$$

$$\therefore\ \frac{r^2 d\omega}{dt} = c.$$

Now $\frac{r^2 d\omega}{2}$ is the differential of the area of a plane curve considered as a spiral; and hence since the ratio of the element of the area to the element of the time is constant, we have the area in any time proportional to that time and equal to $\frac{1}{2}\,ct$.

The two other equations lead to the same result with regard to the areas described by the radius vector of the projection of the point on each of the planes of x, z, and y, z.

Therefore *the projections on the three planes of the areas described by the radius vector of a revolving particle acted on by a force tending to a fixed centre are proportional to the times of their description.*

158. *Cor.* 2. Reciprocally, if these areas are proportional to the times of their description, *the force tends to a fixed centre.*

We have here given the three equations

$$\frac{xdy - ydx}{dt} = c,$$

$$\frac{zdx - xdz}{dt} = c',$$

$$\frac{ydz - zdy}{dt} = c''.$$

Therefore by retracing our steps and differentiating we have

$$\frac{xd^2y - yd^2x}{dt^2} = 0,$$

$$\frac{zd^2x - xd^2z}{dt^2} = 0, \qquad \frac{yd^2z - zd^2y}{dt^2} = 0.$$

Comparing these with the first equations of art. 156, we find

$$xY - yX = 0, \qquad zX - xZ = 0, \qquad yZ - zY = 0,$$

$$\therefore X : Y : Z :: x : y : z.$$

Hence the resultant of X, Y, and Z, must be in the direction of the straight line which joins the position of the particle and the origin of coordinates.

159. If we know that the motion is confined to one plane, the investigation may be made more simple, since the plane may be taken as one of the coordinate planes, as for instance for the plane of x, y. In this case $Z = 0$, and the other two equations are sufficient. The full discussion of an important case of motion in one plane is reserved for a separate chapter. See ch. 2.

160. If the motion is confined to a line, this may be assumed as the axis of x. In this case $Z = 0$, and also $Y = 0$, and the other equation will determine the motion. The discussion of this equation is reserved for chapter 3.

161. If at the same time $X = 0$, $Y = 0$, and $Z = 0$, the three equations

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2z}{dt^2} = 0,$$

give by integration

$$dx = a\,dt, \qquad dy = b\,dt, \qquad dz = c\,dt,$$

a, b, c, being constants. Then integrating again we find

$$\begin{aligned} x &= at + a', \\ y &= bt + b', \\ z &= ct + c', \end{aligned}$$

a', b', c', being new constants introduced by the integration. But if we assume the origin of coordinates to correspond with the origin of the time, we ought to have at once

$$t = 0, \qquad x = 0, \qquad y = 0, \qquad z = 0,$$

and in that case

$$a' = 0, \qquad b' = 0, \qquad c' = 0,$$

therefore $x = at, \qquad y = bt, \qquad z = ct,$

eliminating t between these three equations

$$x = \frac{a}{c} \cdot z, \qquad y = \frac{b}{c} \cdot z,$$

which are the equations to a straight line in space.

It also appears from art. 152, that $v^2 = C$, or the velocity is is constant.

Therefore *a particle in motion, if acted on by no accelerating force, will persevere in uniform rectilinear motion:* which appears also from our hypothesis (147).

CHAPTER II.

ON THE MOTION OF A POINT ACTED ON BY A CENTRAL FORCE.

162. As the case of a point revolving in an orbit by the action of a force tending to a fixed centre is of so much importance, and as the formulæ are capable of considerable simplification when the force is given as a function of the distance from the centre, it has been thought proper to discuss them separately.

We have shewn (156) that the motion will take place in a plane passing through this centre, but this is perhaps apparent, as there is no reason why the point should deviate from it on one side rather than on the other.

Assuming therefore this plane as the plane of x, y, we have only two fundamental equations

$$\frac{d^2x}{dt^2} = \text{X}, \qquad \frac{d^2y}{dt^2} = \text{Y}.$$

But as the force is generally given as a function of the distance from the centre, it is most natural and convenient to introduce polar instead of rectangular coordinates.

163. *Prop. To transform the expressions for the force from rectangular to polar coordinates.*

Suppose the origin to be at the pole or fixed centre.

Let F denote the central force,
r the radius vector or distance of the particle,
ω the angle made by r with the axis of x.

Then $\text{X} = -\text{F}.\cos.\omega$, $\qquad \text{Y} = -\text{F}.\sin.\omega$

the negative sign being introduced because the force tends towards the pole, in opposition to X and Y.

Then multiplying the first of these two equations by sin. ω, and the second by cos. ω, and subtracting, we have

$$X \sin. \omega - Y \cos. \omega = 0.$$

Similarly multiplying the first by cos. ω, and the second by sin. ω, and adding,

$$X \cos. \omega + Y \sin. \omega = - F.$$

But since $x = r \cos. \omega$, and $y = r \sin. \omega$,
we obtain by differentiating each twice

$$d^2x = d^2r \cos. \omega - 2dr\, d\omega \sin. \omega - r d\omega^2 \cos. \omega - r d^2\omega \sin. \omega.$$
$$d^2y = d^2r \sin. \omega + 2dr\, d\omega \cos. \omega - r d\omega^2 \sin. \omega + r d^2\omega \cos. \omega.$$

By dividing these expressions by dt^2 the values of X and Y are obtained, and if we substitute these values in the two expressions obtained above, we have

$$2 \frac{dr}{dt} . \frac{d\omega}{dt} + r . \frac{d^2\omega}{dt^2} = 0 \qquad \text{(A.)}$$

$$\frac{d^2r}{dt^2} - r \frac{d\omega^2}{dt^2} + F = 0 \qquad \text{(B.)}$$

which are two polar equations to determine the motion.

164. *Cor.* 1. If we multiply equation (A.) by $r dt$,

$$2\, r dr . \frac{d\omega}{dt} + r^2 . \frac{d^2\omega}{dt} = 0$$

which is the differential of $\frac{r^2 d\omega}{dt}$. Therefore, integrating,

$$\frac{r^2 d\omega}{dt} = c.$$

and since $\frac{1}{2}\int r^2 d\omega$ = area described by the radius vector, *the areas described by the radius vector of a point revolving round a centre of force, are proportional to the times of their description.*

165. *Cor.* 2. From the last equation we obtain

$$\frac{d\omega}{dt} = \frac{c}{r^2}$$

Hence since the angular velocity of the radius vector is measured by the angle that would be uniformly described in a given time (146), we find that *the angular velocity of the radius vector varies* inversely as the square of the distance.

166. *Prop.* To determine the velocity of the particle in its orbit.

If we multiply equation (A.) by $r\,d\omega$, and equation (B.) by dr, and add the results, we find

$$dr.\frac{d^2 r}{dt^2} + r dr.\frac{d\omega^2}{dt^2} + r^2 d\omega.\frac{d^2\omega}{dt^2} + \mathrm{F}\,dr = 0,$$

$$\text{or } d\left(\frac{dr^2}{dt^2} + r^2.\frac{d\omega^2}{dt^2}\right) + 2\,\mathrm{F}\,dr = 0.$$

Therefore, integrating,

$$\frac{dr^2}{dt^2} + r^2.\frac{d\omega^2}{dt^2} + 2\int \mathrm{F}\,dr = \mathrm{C}$$

and since $v = \frac{ds}{dt}$, and $ds^2 = dr^2 + r^2 d\omega^2$.

$$v^2 = C - 2\int F dr.$$

This expression is analagous to that deduced in art 152, and as it was there shewn that the velocity acquired in passing from a given point to any other point did not depend on the curve described, so the same may be deduced here. And hence also it may be shewn, that if the velocities of two particles moving in any orbit by the action of a central force are equal at one distance, they are equal at all equal distances.

167. Prop. *To find the time of describing any portion of the orbit.*

An expression for the time is easily found from 164, whence

$$dt = \frac{r^2 d\omega}{c}$$

Or by substituting for $\frac{d\omega}{dt}$ in the equation in the last article,

$$\frac{dr^2}{dt^2} + \frac{c^2}{r^2} + 2\int F dr = C,$$

which gives us the relation between r and t, independent of the orbit.

168. Prop. *To find the equation to the orbit.*

Substituting in the expression in the last for dt its value $\frac{r^2 d\omega}{c}$, we have

$$c^2\left(\frac{dr^2}{r^4 d\omega^2} + \frac{1}{r^2}\right) + 2\int F dr = C.$$

This may be put into a more convenient form by assuming

$$u = \frac{1}{r}$$

$$\therefore\ c^2\left(\frac{du^2}{d\omega^2}+u^2\right)-2\int F.\frac{du}{u^2}=C.$$

From this expression we can obtain the relation between u and ω by integration, if the law of the force is given; and *vice versa* from the relation between u and ω, or the equation to the orbit, the law of the force may be found.

169. *Cor.* 1. A very useful expression may be derived from the first form of the last article.

Let p = perpendicular let fall from the centre or pole on the tangent to the point of the orbit, then

$$p^2=\frac{r^4\,d\omega^2}{dr^2+r^2\,d\omega^2}$$

And also $c = 2.$ area in a given time $= 2a$

$$\therefore\ \frac{4a^2}{p^2}+2\int F\,dr=C.$$

Whence by differentiation $F=\dfrac{4a^2\,dp}{p^3\,dr.}$

This expression is most useful in finding the law of force by which a given orbit can be described.

170. *Cor.* 2. Since $v^2=\dfrac{ds^2}{dt^2}=\dfrac{dr^2+r^2d\omega^2}{dt^2}=\dfrac{r^4\,d\omega^2}{p^2\,dt^2}$

$$=\frac{4a^2}{p^2}$$

$$\therefore\ v^2=F.\frac{p\,dr}{dp}=2\,F.\frac{p\,dr}{2\,dp}$$

$= 2$ F. $\frac{1}{4}$ chord of curvature through the centre.

CHAPTER III.

ON THE RECTILINEAR MOTION OF A FREE POINT.

171. If a point at rest is acted on by any single force or forces whose resultant is equivalent to a single force, the motion must be entirely in the direction of the force, and may at once be determined from equations (I.) and (II.), viz.

$$v = \frac{ds}{dt} \text{ and } f = \frac{dv}{dt}.$$

But in order that we may be able to integrate these expressions, the law of the variation of the force must be given as a function either of the space, time, or velocity: and as the intensity of a variable force would depend upon the position of the body, the force is naturally and generally given as a function of the distance from some fixed point.

In order that we may also obtain an exact expression for the value of the force at any variable distance, its value at some given distance must be known. This given distance is most conveniently assumed $= 1$; and if $\phi =$ the intensity of the force at this unit of distance, it is usually called the *absolute* force.

If then the force be given to vary as the n^{th} power of the distance x from a fixed point, we have

$$f = \varphi x^{n}.$$

172. If the point were moving before the application of the force in the line in which the force acts, the motion would still continue rectilinear, whether the original motion were in the

same or opposite direction; and the effect will be the same, however that motion may be supposed to have originated, i. e. whether from any initial velocity communicated to it, or from the action of the force during any previous time.

173. Prop. *To determine the velocity of a particle moving in a right line to or from a fixed point, by the action of a variable force.*

Eliminating $d\,t$ from the equations (I.) and (II.) we have

$$f = \frac{v\,d\,v}{d\,s}$$

or, measuring the line on the axis of x, $f = \frac{v\,d\,v}{d\,x}$.

Hence, by integration,

$$v^2 = \mathrm{C} + 2\int f\,d\,x.$$

And if the force varies as some function of the distance x, the integration may in general be effected.

If the motion is towards the fixed point, $d\,x$ will be *negative*, if from it, *positive*.

The expression here obtained must be compared with that in art. 166.

174. Prop. *To determine the time of the motion.*

Here we have at once from equation (I.)

$$t = \int \frac{d\,x}{v} + \mathrm{C}.$$

And since v is determined in the last article in terms of x, the integration may in general be effected.

175. In the application of these and similar formulæ deduced in the preceding chapters, some other circumstances with regard

to the force, or the motion must be given, in order that we may be able to determine the constants introduced in the integration. Thus in finding the velocity at any point, the velocity at some other known point must be given as in art. 153. Or if the point move from rest, the point at which the motion commences, or when $v = 0$ may be conveniently given.

176. If the force is *uniform,* the formulæ may at once be integrated and present some remarkable results, to exhibit which we will consider the following

Prop. To determine the velocity of a point acted on by an uniform force, in terms of the time of the motion; the velocity of the point in the line of the force being given at the commencement of the time.

Let $v' =$ the given velocity,

Then the velocity due to the action of the force in the time t may be determined from the equation (147)

$$v = ft$$

where f represents the velocity generated by the action of the force in the conventional unit of time, one second.

Hence the whole velocity of the point at the end of the time t is

$$v = v' \pm ft \qquad (a)$$

in which the positive sign must be used if the force is *accelerating,* or in the same direction as v' the initial velocity, and the negative sign if the force is *retarding,* or in the contrary direction.

We may here observe that the same force may be an accelerating or a retarding force, according as it acts in the same or contrary direction to the motion of the particle.

177. *Prop. The same things being given, to determine the space in terms of the time.*

By equation (I.)

$$ds = v\,dt,$$

Therefore substituting the value of v found in the last article,

$$ds = v'dt + ft\,dt.$$

By integration we obtain

$$s = c + v't \pm \tfrac{1}{2}ft^2, \qquad (b)$$

in which c is an arbitrary constant depending on the position of the point when $t = 0$.

178. *Cor.* 1. If the motion of only one point is under consideration, it is most convenient to reckon the space from the position of the point when $t = 0$, or at the origin of the time.

In this case $c = 0$, whence we obtain

$$s = v't \pm \tfrac{1}{2}ft^2. \qquad (c)$$

179. *Cor.* 2. If the point move from rest, $v' = 0$, and we have

$$s = \tfrac{1}{2}ft^2. \qquad (d)$$

180. *Prop. The same things being given, to determine the velocity generated in terms of the space described.*

By eliminating t between equations *(a)* and *(c)* we find

$$2fs = v^2 \backsim v^2. \qquad (e)$$

181. *Cor.* 1. If the point move from rest, then

$$v^2 = 2fs. \qquad (f)$$

182. *Cor.* 2. If we eliminate f between the equations *(d)* and *(f)* we have

$$s = \tfrac{1}{2} t v.$$

Therefore the space described from rest by a point acted upon by an uniform force to acquire a certain velocity, is half the space due to the same velocity continued uniformly (146).

183. The formulæ here deduced to acquire velocity through any space, or during any time, are applicable as well to cases where the motion is opposite to the force; as, for instance, to the case where the point moves until the whole velocity is destroyed, &c.

CHAPTER IV.

ON THE CONSTRAINED MOTION OF A POINT.

184. If a point is constrained to move on a curve, a new force of reaction is introduced, and we may proceed to investigate its effects as a force independent of the curve, as in Statics (19).

Prop. To determine the velocity of a point moving on a given curve.

Let N be the normal force of reaction,

α, β, γ, the angles which the normal makes with the axes of coordinates:

and let X, Y, Z be the sums of the other forces resolved according to the three axes as before; and if N be similarly resolved, we have by the equations (V.)

$$\frac{d^2x}{dt^2} = \text{N} \cos. \alpha + \text{X},$$

$$\frac{d^2y}{dt^2} = \text{N} \cos. \beta + \text{Y},$$

$$\frac{d^2z}{dt^2} = \text{N} \cos. \gamma + \text{Z}.$$

Also since we have, as in art. 19,

$$\frac{dx}{ds} \cos. \alpha + \frac{dy}{ds} \cos. \beta + \frac{dz}{ds} \cos. \gamma = 0.$$

by multiplying the first of the three equations by $2dx$, the second by $2dy$, and the third by $2dz$, and adding, we obtain

$$\frac{2dxd^2x + 2dyd^2y + 2dzd^2z}{dt^2} = 2(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz).$$

The first member of this equation is the exact differential

$$\text{of } \frac{dx^2 + dy^2 + dz^2}{dt^2} = \frac{ds^2}{dt^2} = v^2.$$

Whence by integration

$$v^2 = 2\int(\mathrm{X}dx + \mathrm{Y}dy + \mathrm{Z}dz) + \mathrm{C}.$$

185. This result is exactly the same as that obtained for the unconstrained motion of a point deduced in art. 152, whence we may conclude,

First, *That if no accelerating force acted on the point, its velocity would remain constant, and not at all retarded by the action of the curve.* Compare art. 152.

Secondly, *That if any accelerating forces do act on the point, the velocity is independent of the curve on which it is compelled to move.* Compare art. 153.

186. *Cor.* 1. If the forces are all in one plane, this may be taken as the plane of x, y, and two of the original equations are sufficient. Here

$$v^2 = 2\int(\mathrm{X}dx + \mathrm{Y}dy) + \mathrm{C}.$$

187. *Cor.* 2. If the forces are all parallel to one another, they may be assumed parallel to the axis of x, and in this case

$$v^2 = 2\int\mathrm{X}dx + \mathrm{C}.$$

188. *Cor.* 3. If in this last case the force, or resultant of the forces is *uniform*, let it $= f$,

$$\therefore\ v^2 = \mathrm{C} \pm 2fx,$$

where the upper or lower sign is to be used according as the force tends to increase or decrease x.

If a be assumed to the distance of the point on axis of x where the motion commences, or $v = 0$,

$$\therefore\ v^2 = 2f(a - x).$$

And since only the ordinates on the axis of x are involved, the velocity of the point on the curve does not at all depend on the curve described, but on the difference of the ordinates on axis of x.

189. *To find the time of a point moving on a given curve.*

In all cases $dt = \frac{ds}{v}$.

Hence if the nature of the curve and the velocity at any point be known, the time may be found by the proper substitutions and integration.

190. *Cor.* If the forces act in parallel lines, and their resultant is *uniform*, as in art. 188, we have

$$dt = \frac{ds}{\sqrt{\left(2f(a - x)\right)}}$$

191. If the force tends to a fixed centre it will be most convenient to introduce polar coordinates as in chapter II, and by similar substitutions we obtain

$$v^2 = C - 2\int F\,dr.$$

$$dt = \frac{ds}{\sqrt{(C - 2\int F\,dr)}}.$$

192. Prop. *To find the magnitude of the reaction of the curve.*

The most obvious way of determining the whole pressure on the curve, is to find the value of N from the three equations in art. 184; but the number of terms obtained by this process renders it very laborious. We may however first suppose that no accelerating force acts on the point and we then have

$$\frac{d^2x}{dt^2} = \text{N cos. } \alpha, \quad \frac{d^2y}{dt^2} = \text{N cos. } \beta, \quad \frac{d^2z}{dt^2} = \text{N cos. } \gamma,$$

$$\therefore \text{N} = \frac{\sqrt{\left((d^2x)^2 + (d^2y)^2 + (d^2z)^2\right)},}{dt^2}$$

And eliminating dt by means of the equation $v = \frac{ds}{dt}$

$$\text{N} = v^2. \ \frac{\sqrt{\left((d^2x)^2 + (d^2y)^2 + (d^2z)^2\right)},}{ds^2}$$

but if ρ be the radius of the osculating circle[a]

$$\rho = \frac{ds^2}{\sqrt{\left((d^2x)^2 + (d^2y)^2 + (d^2z)^2\right)},}$$

Therefore $\text{N} = \frac{v^2}{\rho}$.

We thus obtain the pressure which the point exerts against the curve from its own inertia.

But if it is acted on by any accelerating forces, we must resolve them into two directions, one perpendicular to the curve, and the other in the direction of a tangent; then the *sum* of the perpendicular forces will give the pressure due to the action of the forces. If we then add this pressure to that which we have investigated as due to the inertia of the particle, we shall obtain the whole pressure.

---

[a] See Lardner's Differential Calculus, 186.

193. If the motion is in one plane, the investigation is much more simple.

For if we assume this plane as the plane of x, y,

$$\therefore\ z = 0, \qquad Z = 0, \qquad \text{and } \cos.\gamma = 0,$$

and the two first equations (184) are sufficient.

Also we have $\cos.^2\alpha + \cos.^2\beta = 1$

$$\text{and } \frac{dx}{ds}\cos.\alpha + \frac{dy}{ds}\cos.\beta = 0.$$

Whence we find

$$\cos.\alpha = \pm\frac{dy}{ds}, \qquad \cos.\beta = \mp\frac{dx}{ds},$$

in which the upper signs must be taken together, and so also the lower signs give corresponding values. The value of N itself must always be positive, and we can therefore ascertain which signs must be used.

If we substitute these values in the two equations (184) we have

$$\frac{d^2x}{dt^2} = \mathrm{X} \pm \mathrm{N}.\frac{dy}{ds},$$

$$\frac{d^2y}{dt^2} = \mathrm{Y} \mp \mathrm{N}.\frac{dx}{ds}.$$

Then multiplying the first by $\frac{dy}{ds}$ and the second by $\frac{dx}{ds}$, we obtain, by subtraction,

$$N = Y\frac{dx}{ds} - X\frac{dy}{ds} + \frac{dy\,d^2x - dx\,d^2y}{ds\,dt^2},$$

Eliminating dt by means of the equation $v = \frac{ds}{dt}$,

$$N = Y\frac{dx}{ds} - X\frac{dy}{ds} + v^2.\frac{dy\,d^2x - dx\,d^2y}{ds^3},$$

but if ρ be the radius of the osculating circle,

$$\rho = \frac{ds^3}{dy\,d^2x - dx\,d^3y},$$

Therefore $N = Y\frac{dx}{ds} - X\frac{dy}{ds} + \frac{v^2}{\rho}$.

Of the two parts of this expression the first is exactly equal and opposite to that which would be obtained by resolving X and Y according to the normal to the curve. We therefore have in this expression that part of the pressure on the curve which arises from the forces which act on the particle.

The latter part, which is the same as that deduced in the last article, is that part of the pressure which arises from the motion or inertia of the particle.

193. *Cor.* 1. If the accelerating forces act in parallel lines, as in art. 187,

$$\therefore N = -X\frac{dy}{ds} + \frac{v^2}{\rho}.$$

194. *Cor.* 3. If the force is *uniform*, let it $= f$

$$\therefore N = -f\frac{dy}{ds} + \frac{v^2}{\rho}.$$

195. *Cor.* 4. If the particle move in a straight line,

$$\therefore \rho = \infty$$

and the latter part of the expression disappears.

196. If we introduce polar coördinates we have

$$N = F. \frac{r\,d\omega}{ds} + \frac{v^2}{\rho}.$$

197. The expression $\frac{v^2}{\rho}$ is the measure of the pressure on the curve arising from the tendency of the particle to move in a right line, a tendency which is counteracted, though not destroyed, by the normal forces. This force or pressure being always directed from the centre of the circle of curvature, is called the *centrifugal force.*

198. If the particle be constrained to move on a circle, the radius of curvature is the radius of the circle, and therefore if $f_{,}$ represent the centrifugal force, and r = radius of circle,

$$f_{,} = \frac{v^2}{r}.$$

Or if, instead of supposing it inclosed within a circle, we conceive it attached to the extremity of an inextensible thread, and to revolve round a fixed point so as to describe a circle, the tension which the thread will undergo from the tendency to persevere in a right line will replace the pressure of the point against the curve arising from the same tendency, and consequently will be measured by the same expression $\frac{v^2}{r}$.

Or if, instead of the thread, we suppose it retained in the circular orbit, or indeed any other orbit whatsoever, by any attractive force tending to that centre, the resistance opposed to the attractive force, or the force which prevents it from falling into the centre, is measured by $\frac{v^2}{\rho}$.

199. If the motion be circular and uniform, we have

if T = time of one revolution,
$2\pi r$ = the whole circumference,

$$\text{since (146) } s = t v, \quad \therefore v = \frac{2\pi r}{T}.$$

Hence, by substituting for v in the expression for the centrifugal force,

$$f_{,} = \frac{4\pi^2 r}{T^2},$$

which is the most convenient expression for finding the absolute value of the centrifugal force.

CHAPTER V.

ON THE MOTION OF A SYSTEM IN GENERAL.

200. The conclusions hitherto deduced are applicable generally only to the motion of a point or particle, though in certain cases to the motion of a system or assemblage of particles. For instance, if the several particles are equally accelerated, and move in parallel directions, the deductions respecting the path, velocity, or time of one point, will evidently apply to the system.

But yet with regard to the forces which produce the motion in such a case, and in the case of a single particle, very different considerations are involved. In the motion of a single particle, the accelerating force was measured by the velocity generated in the unit of time. Here we have a system in which every particle is impressed with the same velocity, and consequently the whole system must be acted on by a force equal to the sum of the forces which act on each particle. If we call, therefore, the *number* of particles the *mass* of the system, the *force would be measured by the product of the mass and the velocity.*

If the system move in any variable manner, as the accelerating force would be measured by $\frac{dv}{dt}$ (147), so the force on the whole system would be measured by

$$m. \frac{dv}{dt},$$

if $m =$ mass or number of particles of the system.

This product of the mass and the velocity of each particle is called the *momentum*, or more properly the *quantity of motion*.

The force generating this quantity of motion is called the *moving force,* in opposition to the term *accelerating* force, which is only applied to that which acts on the unit of mass.

201. Since then every system or mass is considered as composed of perfectly *inert* particles (147), if the system is in motion, we have at any instant an uniform rectilinear motion impressed upon each particle which it endeavours to maintain. These motions are parallel to each other; and since their number and the *ratio* of their intensities remain the same, whether the rate of the motion be greater or less, we have in every system a point analogous to the *centre of parallel forces,* which we here call the *centre of inertia*[a]. Even if the system be not in motion we may consider this point as existing, inasmuch as the property of inertia attaches to rest or to motion, and as it would occupy a certain fixed position were any motion communicated. Its position may be determined by the formulæ for the determination of the centre of parallel forces (24), and since the force of inertia is proportional to the mass or number of particles of each part, we have

$$x_{,} = \frac{\Sigma.\, m x}{\Sigma.\, m}, \qquad y = \frac{\Sigma.\, m y}{\Sigma.\, m}, \qquad z_{,} = \frac{\Sigma.\, m z}{\Sigma.\, m}.$$

202. If we consider the motion of a system whose parts are connected, we perceive that it is not only influenced by the force which has acted or still acts on it, but that it is affected by the mutual action of its several parts: and that the system will seldom move in the same manner, as if it were free to obey the action of the force. In order, therefore, to determine the motion in these cases, we have need of some other principles of investi-

---

[a] This point is usually called the centre of gravity of the system, but the term *gravity* is manifestly misapplied.

gation, and we will introduce a principle which is due to d'Alembert, which may be thus enunciated.

If several forces are applied to the different parts of a system, and if these forces ought to impress certain motions on the parts, we may consider each of these impressed motions *as compounded of two, one* actual, *and the other* destroyed.

203. *Cor.* It is evident that a system influenced only by these latter, ought to remain in equilibrio.

204. This principle may be put into a more convenient form thus:

Prop. *If a motion is impressed on the several parts of the system, equal but contrary to those which it ought to receive, the system will be at rest,* or, in other words, there will be an equilibrium between the *impressed* and *actual* motions, if they be supposed to act in directions contrary to each other.

For if m', m'', m''', &c. represent the masses of the parts,

v', v'', v''', &c. the *impressed* velocities,

Let v' be supposed to be resolved into u' and $u_{,}$, the first of which represents the *actual* velocity of the part, and the other the velocity *destroyed;* and let v'', v''', &c. be supposed to be similarly resolved.

Then the motions which the parts would have received if free, or the *impressed* motions,

$$m'v', \; m''v'', \; m'''v''', \; \&c.$$

are equivalent to

the *actual* motions, $m'u'$, $m''u''$, $m'''u'''$, &c.

and the *destroyed,* $m'u_{,}$, $m''u_{,,}$, $m'''u_{,,,}$, &c.

which last are (203) in equilibrio with each other.

Hence since $m'v'$ is the resultant of $m'u'$ and $m'u_{,}$

$\therefore$ $m'u_{,}$ is the resultant of $m'u'$ and $-m'v'$

So $m''u_{,,}$ is the resultant of $m''u''$ and $-m''v''$

&c. &c.

and since the resultants are in equilibrio, the components must be so also.

205. The method here introduced consists in substituting for the forces acting on a system, two sets of equivalent forces. Those of the first set impart such motions as each particle may freely follow, without causing any impediment to the rest, or being at all impeded by them: those of the second set, on the contrary, are counteracted by the mutual impediments and are in equilibrio with each other. The first are fully effective, and determine the actual motion of the system; the second are wholly destroyed, and determine the pressure which each point suffers from the mutual action of the parts.

CHAPTER VI.

ON THE MOTION OF ANY SYSTEM.

206. Prop. *To determine the motion of a system whose parts act on each other in any arbitrary manner, and are acted on by any accelerating forces.*

Let m', m'' m''', &c. be the masses of the parts

x', y', z', the cordinates of m'

x'', y'', z'' m''

&c. &c.

Then if X', Y', Z', are the *accelerating* forces which act on m', resolved parallel to the three rectangular axes,

$m'X'$, $m'Y'$, $m'Z'$ are the *moving forces.*

The velocity due to the force X' in any time dt, will be (147) $X'dt$, and that due to the inertia of the part at the end of any instant will be in the same axis $\frac{dx'}{dt}$; and similarly in the other axes.

Hence the *impressed* motions are

$$m'\left(\frac{dx'}{dt} + X'dt\right),\ m'\left(\frac{dy'}{dt} + Y'dt\right),\ m'\left(\frac{dz'}{dt} + Z'dt\right).$$

But if the velocity in the axis of x at any time t is $\frac{dx'}{dt}$, at the end of the time $t + dt$ it will be $\frac{dx'}{dt} + d.\frac{dx'}{dt}$, and similarly for the other axes;

Hence the *effective* motions are

$$m'\left(\frac{dx'}{dt}+\frac{d^2x'}{dt}\right),\quad m'\left(\frac{dy'}{dt}+\frac{d^2y'}{dt}\right),\quad m'\left(\frac{dz'}{dt}+\frac{d^2z'}{dt}\right).$$

Considering, therefore, the impressed motions as acting in the contrary directions, and therefore as negative, we obtain for the force acting on the mass m',

$$m'\left(\frac{d^2x'}{dt}-\mathrm{X}'dt\right),\quad m'\left(\frac{d^2y'}{dt}-\mathrm{Y}'dt\right),\quad m'\left(\frac{d^2z'}{dt}-\mathrm{Z}'dt\right).$$

If we mark the accented letters first with two and then with three accents, and so on, we shall obtain similar expressions for the forces acting on the masses m'', m''', &c.

But by the principle (204), the system is in equilibrio by the action of these forces, we have therefore to substitute these values of the forces in the six general equations of equilibrium deduced in art. 33, in order to express this condition. For this we must observe that $p'\cos.\alpha'$, $p'\cos.\beta'$, $p'\cos.\gamma'$, are equivalent to the three expressions obtained above for the force acting on m'; and we therefore obtain from the first,

$$m'\left(\frac{d^2x'}{dt}-\mathrm{X}'dt\right)+m''\left(\frac{d^2x''}{dt}-\mathrm{X}''dt\right)+\text{\&c.}=0,$$

$$\text{or } \Sigma\, m.\frac{d^2x}{dt^2}=\Sigma.\, m\mathrm{X};$$

and similarly from the two next

$$\Sigma.\, m.\frac{d^2y}{dt^2}=\Sigma.m\mathrm{Y},\qquad \Sigma.\, m.\frac{d^2z}{dt^2}=\Sigma.m\mathrm{Z}.$$

The other three equations, L, M, N, give us by the same substitutions,

$$\Sigma.\, m\left(\frac{xd^2y - yd^2x}{dt^2}\right) = \Sigma.\, m\,(xY - yX),$$

$$\Sigma.\, m\left(\frac{zd^2x - xd^2z}{dt^2}\right) = \Sigma.\, m\,(zX - xZ),$$

$$\Sigma.\, m\left(\frac{yd^2z - zd^2y}{dt^2}\right) = \Sigma.\, m\,(yZ - zY).$$

Such are the equations which will determine the motion of any free system. But even if any part of the system is constrained to move on a given curve, we have only to introduce the resistance of the curve as a new force (183) and then we may consider the system as entirely free.

These six equations involve some important principles which we will proceed to develop: and we will first discuss the first three, which enable us to discover the motion of the *centre* of inertia of the system.

207. PROP. *To determine the motion of the centre of inertia of any system.*

If $x_{,}$ $y_{,}$ $z_{,}$ are the coordinates of the centre of inertia, we have (201)

$$x_{,} = \frac{\Sigma.\, m\, x}{\Sigma.\, m}, \qquad y_{,} = \frac{\Sigma.\, m\, y}{\Sigma.\, m}, \qquad z_{,} = \frac{\Sigma.\, m\, z}{\Sigma.\, m}.$$

Differentiating these expressions twice with respect to t, we find

$$\frac{d^2x_{,}}{dt^2} = \frac{\Sigma.\, m.\, \frac{d^2x}{dt^2}}{\Sigma.\, m}, \qquad \frac{d^2y_{,}}{dt^2} = \frac{\Sigma.\, m.\, \frac{d^2y}{dt^2}}{\Sigma.\, m},$$

$$\frac{d^2z_{,}}{dt^2} = \frac{\Sigma.\, m.\, \frac{d^2z}{dt^2}}{\Sigma.\, m}.$$

Hence by the first three of the general equations (205)

$$\frac{d^2 x_{,}}{dt^2} = \frac{\Sigma.\, m\, X}{\Sigma.\, m}, \quad \frac{d^2 y_{,}}{dt^2} = \frac{\Sigma.\, m\, Y}{\Sigma.\, m}, \quad \frac{d^2 z_{,}}{dt^2} = \frac{\Sigma.\, m\, Z}{\Sigma.\, m}.$$

But if $X_{,}$, $Y_{,}$, $Z_{,}$ be taken to represent the three components of the resultant of all the accelerating forces which act on the system, and M the whole mass of the system, or $M = \Sigma.\, m$.

$$\therefore\ M X_{,} = \Sigma.\, m X, \quad M Y_{,} = \Sigma.\, m Y, \quad M Z_{,} = \Sigma.\, m Z.$$

Whence we obtain

$$\frac{d^2 x_{,}}{dt^2} = X_{,}, \quad \frac{d^2 y_{,}}{dt^2} = Y_{,}, \quad \frac{d^2 z_{,}}{dt^2} = Z_{,}$$

which are the three equations for the motion of a point (151), acted on by the accelerating forces $X_{,}$, $Y_{,}$, $Z_{,}$ in each axis.

Hence we conclude that the *centre of inertia of a system will move as if the whole mass were there united, and a moving force equal to the resultant of all the forces were immediately applied to it.*

208. If no accelerating force acts upon the system, but the parts are merely influenced by their mutual action, let F represent the action of one part of the mass m' upon another m''.

Then m' F is the force with which m'' is influenced by m', and similarly m'' F is that with which m' is influenced by m''.

If then x', y', z' are the coordinates of m',

x'', y'', z'' of m'',

and r = the distance of the two,

by resolving the forces, and calling X′, Y′, Z′ the components of the force by which m'' is acted on, we have

$$X' = m''F.\frac{x'' - x'}{r}, \qquad Y' = m''F.\frac{y'' - y'}{r},$$

$$Z' = m''F.\frac{z'' - z'}{r};$$

so also if X″, Y″, Z″ represent the components of the force which acts on m',

$$X'' = m'F.\frac{x' - x''}{r}, \qquad Y'' = m'F.\frac{y' - y''}{r},$$

$$Z'' = m'F.\frac{z' - z''}{r}.$$

Hence

$$m'X' + m''X'' = 0,$$

and also

$$m'Y' + m''Y'' = 0, \qquad \text{and } m'Z' + m''Z'' = 0.$$

The same may be proved for the mutual action of any other parts of the mass, and therefore

$$\Sigma . mX = 0, \qquad \Sigma . mY = 0, \qquad \Sigma . mZ = 0.$$

Hence we have from the last article

$$\frac{d^2x_,}{dt^2} = 0, \qquad \frac{d^2y_,}{dt^2} = 0, \qquad \frac{d^2z_,}{dt^2} = 0.$$

Therefore, integrating twice we obtain,

$$x_, = at + a', \qquad y_, = bt + b', \qquad z_, = ct + c',$$

a, b, c, and a', b', c', being constants introduced in the integration.

By eliminating t we find

$$x_, - a' = \frac{a}{c}(z' - c'),$$

$$y_{,} - b' = \frac{b}{c}(z_{,} - c').$$

And since these are the equations of a straight line in space, we conclude that the *motion of the centre of inertia of a system only exposed to the mutual action of its parts will be rectilinear.*

Also, since

$$v^2 = \frac{ds^2}{dt^2} = \frac{dx_{,}^2}{dt^2} + \frac{dy_{,}^2}{dt^2} + \frac{dz_{,}^2}{dt^2},$$

we find

$$v^2 = a^2 + b^2 + c^2.$$

Hence the velocity is constant, and the *motion is not only rectilinear, but uniform.*

As, therefore, by the assumed property of the separate particles (147), a material point cannot of itself alter any impression which has been communicated to it, so also a system of particles cannot alter the motion of its centre of inertia, by the mutual action of its own parts.

This remarkable result constitutes a general law, which is denominated the principle of the *Conservation of the centre of Inertia.*

209. Prop. *To determine the motion of the other parts of the system.*

The first three equations in the general proposition (206) have enabled us to determine the motion of the centre of inertia of the system; we next proceed to discuss the three other equations, viz.

$$\Sigma . m\left(\frac{x d^2 y - y d^2 x}{dt^2}\right) = \Sigma . m (x\text{Y} - y\text{X}),$$

$$\Sigma . m \left(\frac{z d^2 x - x d^2 z}{d t^2}\right) = \Sigma . m (z X - x Z),$$

$$\Sigma . m \left(\frac{y d^2 z - z d^2 y}{d t^2}\right) = \Sigma . m (y Z - z Y).$$

Here we may observe that the second members of these equations will each become = 0,

First. If there is no accelerating force acting on the system.

Secondly. If the accelerating force tends towards the origin of coordinates;

For let F represent the force acting on any part, and tending towards the origin,

and r = the distance of that point from the origin;

Then, by resolving F, as in art. 157, we have

$$X = F . \frac{x}{r}, \quad Y = F . \frac{y}{r}, \quad Z = F . \frac{z}{r}.$$

If we substitute these values in the expressions

$$x Y - y X, \quad z X - x Z, \quad y Z - z Y,$$

they each become = 0; and the same may be shewn for any other force F′, F″, &c. acting on any other parts of the system m', m'', &c.

Thirdly. If the particles are only exposed to their own mutual action.

For if F represent the action of one part of the mass m' upon another m'',

and r = the distance of the parts from each other,

Then by resolving, as in art. 208, the whole accelerating forces in the axis of x will be

$$F . \frac{x' - x}{r} + F . \frac{x - x'}{r} = 0,$$

and those in the axis of Y

$$\text{F}.\frac{y'-y}{r}+\text{F}.\frac{y-y'}{r}=0,$$

and since in this case

$$\Sigma.\,m.\,(x\text{Y}-y\text{X})=m'm''(x\text{Y}-y\text{X}),$$

we have here also the second member of the first of the three equations $= 0$.

And so also for any other particles acting on each other.

The same may be proved in an exactly similar manner for the second members of the other two equations; therefore by integration we obtain

$$\Sigma.\,m.\left(\frac{x\,dy-y\,dx}{dt}\right)=c',$$

$$\Sigma.\,m.\left(\frac{z\,dx-x\,dz}{dt}\right)=c'',$$

$$\Sigma.\,m.\left(\frac{y\,dz-z\,dy}{dt}\right)=c'''.$$

Now the differential expression $x\,dy-y\,dx$ represents the double of the area described round the origin during the instant dt by the projection of the radius vector of the particle, on the plane of x, y; and similarly the expressions $z\,dx-x\,dz$, $y\,dz-z\,dy$, are the doubles of the areas described in the same time by the projections of the same radius on the planes of z, x and y, z;

Therefore the numerators of the first equations represent the sum of the areas traced by the projection of the radii vectores of the different parts of the system on each of the coordinate planes, multiplied respectively by the masses of the parts; this sum is consequently proportional to the element of the time, and in a finite time is proportional to the time.

This remarkable result constitutes another general law which is denominated *the principle of the Conservation of areas.*

210. Prop. *To determine the plane on which the sum of the projected areas is a maximum.*

The property relating to the areas described by the projections of the radii vectores of the several parts of the system holds good for any plane through the origin. The sum of the areas will evidently vary with the position of the plane, and there will be one plane on which the sum of these projections, multiplied by their masses, will be a maximum.

To determine this plane,

let α', α'', α''', be the angles which it makes with the three coordinate planes;

A = sum of all the areas traced on it by the projections of the radii and multiplied by the masses.

We have then, by the property of projections[a],

$$\Sigma.\, m. \left(\frac{xdy - ydx}{dt}\right) = \text{A}.\cos.\,\alpha',$$

$$\Sigma.\, m. \left(\frac{zdx - xdz}{dt}\right) = \text{A}.\cos.\,\alpha'',$$

$$\Sigma.\, m. \left(\frac{ydz - zdy}{dt}\right) = \text{A}.\cos.\,\alpha''',$$

but the first members of these equations are (209) constant,

---

[a] See Hymer's Analytic Geometry, section ii. Or Hamilton's, part. ii. section i. chapter vii.

$$\therefore \text{ A. cos. } \alpha' = c', \quad \text{A. cos. } \alpha'' = c'', \quad \text{A. cos. } \alpha''' = c'''.$$

Hence $$A^2 = c'^2 + c''^2 + c'''^2,$$

and $$\cos. \alpha' = \frac{c'}{\sqrt{c'^2 + c''^2 + c'''^2}},$$

$$\cos. \alpha'' = \frac{c''}{\sqrt{c'^2 + c''^2 + c'''^2}}, \quad \cos. \alpha''' = \frac{c'''}{\sqrt{c'^2 + c''^2 + c'''^2}}.$$

The angles α', α'', α''', are therefore constant, and the principal plane of projection remains always parallel to itself during the whole motion, whatever changes may take place in the position of the various parts of the system.

This plane is from this property called the *invariable plane*[b].

211. Prop. *To determine the position of this invariable plane at any instant.*

For this purpose we must know the coordinates of the several parts of the system and their velocities at that instant. For the position depends on the three constants c', c'', c''', which were found to be (209)

$$c' = \Sigma. m. \left(x \frac{dy}{dt} - y. \frac{dx}{dt}\right),$$

$$c'' = \Sigma. m. \left(z \frac{dx}{dt} - x \frac{dz}{dt}\right),$$

$$c''' = \Sigma. m. \left(y \frac{dz}{dt} - z \frac{dy}{dt}\right).$$

---

[b] This plane, the discovery of which we owe to Laplace, may be of the greatest importance in investigations connected with the system of the world; it presents us with a plane to which the position of the various bodies of the system may be referred at any period.

And if the velocities, viz. $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, and the coordinates x, y, z, at any instant, for all the parts of the mass are given, the values of the three constants are given also.

212. The consideration of the invariable plane is of great advantage in the investigation of dynamical problems. For if it be assumed as one of the coordinate planes, two of the three constants will $= 0$, since $\cos. \alpha'' = \cos. \alpha''' = \cos. \frac{\pi}{2} = 0$. Hence we also conclude that the sum of the projections on every plane perpendicular to the invariable plane $= 0$.

213. *Prop. To determine the motion of any system, if the origin of coordinates have an uniform rectilinear motion.*

We have hitherto considered the origin of coordinates as a fixed point, to which the position of the different parts of the system might be referred; but the six general equations will still hold if we give the origin an uniform rectilinear motion.

For let x, y, z, be the coordinates of this moving origin referred to some fixed point which may be considered as the origin of the coordinates x, y, z, of any part of the system m'.

Let x', y', z', be the coordinates of the same part referred to the moving origin,

$$\therefore\ x = \mathrm{x} + x, \qquad y = \mathrm{y} + y', \qquad z = \mathrm{z} + z'.$$

If we differentiate these equations twice, and substitute the values thus obtained in the first three of the general expressions (206), we find

$$\Sigma.\, m.\left(\frac{d^2\mathrm{x} + d^2x'}{dt^2}\right) = \Sigma.\, m\mathrm{X},$$

$$\Sigma.\,m.\left(\frac{d^2\mathrm{y} + d^2y'}{dt^2}\right) = \Sigma.\,m\,Y,\ \Sigma m.\left(\frac{d^2\mathrm{z} + d^2z'}{dt^2}\right) = \Sigma.\,mZ.$$

But since the motion of the new origin is uniform and rectilinear, we have (161)

$$\frac{d^2\mathrm{x}}{dt^2} = 0, \qquad \frac{d^2\mathrm{y}}{dt^2} = 0, \qquad \frac{d^2\mathrm{z}}{dt^2} = 0.$$

Hence the equations become

$$\Sigma.\,m\,\frac{d^2x'}{dt^2} = \Sigma.\,m\,X, \qquad \Sigma.\,m.\,\frac{d^2y'}{dt^2} = \Sigma.\,m\,Y,$$

$$\Sigma.\,m.\,\frac{d^2z'}{dt^2} = \Sigma.\,m\,Z,$$

which are of exactly the same form as the original ones in art. 206.

By making similar substitutions in the first of the three next equations we have

$$\frac{\mathrm{x}.\,\Sigma.\,m\,d^2y' - \mathrm{y}.\,\Sigma.\,m\,d^2x'}{dt^2} + \Sigma.\,m\left(\frac{x'd^2y' - y'd^2x'}{dt^2}\right)$$

$$= \Sigma.\,m\,(x'Y - y'X) + \mathrm{x}.\,\Sigma.\,m\,Y - \mathrm{y}.\,\Sigma.\,mX.$$

If we compare this with the three equations just deduced, we find that the first part of the first member is equal to the latter part of the second, hence there results

$$\Sigma.\,m.\left(\frac{x'd^2y' - y'd^2x'}{dt^2}\right) = \Sigma.\,m\,(x'Y - y'X).$$

Similarly for the remaining two we have

$$\Sigma.\, m \left(\frac{z'\,d^2x' - x'\,d^2z'}{dt^2}\right) = \Sigma.\, m\, (z'\mathrm{X} - x'\mathrm{Z}),$$

$$\Sigma.\, m \left(\frac{y'\,d^2z' - z'\,d^2y'}{dt^2}\right) = \Sigma.\, m\, (y'\mathrm{Z} - z'\mathrm{Y}).$$

The six general equations therefore preserve exactly the same form, whether we suppose the origin fixed or endued with an uniform rectilinear motion; and hence the general principles before deduced apply to this case also.

214. With respect to the position of the *invariable plane* on this supposition, we may at once observe, that since the motion of the origin is in a right line, we have

$$\mathrm{x}\,d\mathrm{y} - \mathrm{y}\,d\mathrm{x} = 0,$$

$$\mathrm{z}\,d\mathrm{x} - \mathrm{x}\,d\mathrm{z} = 0, \qquad \mathrm{y}\,d\mathrm{z} - \mathrm{z}\,d\mathrm{y} = 0,$$

or the projection of the areas of this moving point on the three planes are each $= 0$.

If we therefore substitute for x, y, z, in the equations in art. 209 we find,

$$\Sigma.\, m. \left(\frac{x'dy' - y'dx'}{dt}\right) = c',$$

$$\Sigma.\, m. \left(\frac{z'dx' - x'dz'}{dt}\right) = c'',$$

$$\Sigma.\, m. \left(\frac{y'dz' - z'dy'}{dt}\right) = c''',$$

in which c', c'', c''', are constants, as before.

But these three constants determine the position of the plane, whence we conclude that this plane will always continue parallel to itself during the rectilinear motion of the origin of coordinates.

215. Hence also, if the origin of coordinates be assumed to coincide with the centre of inertia, the principle of the conservation of areas will exist with reference to this origin, and the invariable plane always passing through this point will be carried along always parallel to itself during the motion of the system.

CHAPTER VII.

ON THE MOTION OF A RIGID SYSTEM, PERFECTLY FREE.

216. Prop. *To determine the motion of a system whose parts are connected in an invariable manner, and acted on by any accelerating forces.*

The six general equations deduced in the last chapter (206) for the motion of any system, whose parts are connected in any arbitrary manner, are easily extended to a rigid system, or one whose parts are connected in an invariable manner. We have in this case merely to substitute for the masses m', m'', m''', &c. the elements of the mass. Let dm be an element of the mass,

X, Y, Z, its coordinates,

x, y, z, the forces acting on it in the three axes.

Then introducing the sign of integration instead of the sign Σ, the equations become

$$\int\frac{d^2x}{dt^2}.\,dm = \int X dm, \qquad \int\frac{d^2y}{dt^2}.\,dm = \int Y dm,$$

$$\int\frac{d^2z}{dt^2}.\,dm = \int Z dm,$$

$$\int\left(\frac{x\,d^2y - y\,d^2x}{dt^2}\right) dm = \int (xY - yX)\,dm,$$

$$\int\left(\frac{z\,d^2x - x\,d^2z}{dt^2}\right)dm = \int (zX - xZ)\,dm,$$

$$\int\left(\frac{y\,d^2z - z\,d^2y}{dt^2}\right)dm = \int (yZ - zY)\,dm.$$

By these six equations we can determine the motion of any rigid system. The last three present us with the principle of areas, and were there any fixed point in the system, would enable us completely to ascertain its movement of rotation round that point.

217. **Prop.** *To investigate the expressions, when the centre of inertia is assumed to be the origin.*

Let $x_{,}$, $y_{,}$, $z_{,}$ be the coordinates of the centre of inertia referred to the first origin,

x', y', z', the new coordinates of the element dm,

we then have

$$x = x_{,} + x', \qquad y = y_{,} + y', \qquad z = z_{,} + z'.$$

If we then substitute these values and their differentials in the first three of the general equations, we obtain

$$\int\left(\frac{d^2x_{,}}{dt^2} + \frac{d^2x'}{dt^2}\right)dm = \int X\,dm,$$

$$\int\left(\frac{d^2y_{,}}{dt^2} + \frac{d^2y'}{dt^2}\right)dm = \int Y\,dm,$$

$$\int\left(\frac{d^2z_{,}}{dt^2} + \frac{d^2z'}{dt^2}\right)dm = \int Z\,dm.$$

But since $x_{,}$, $y_{,}$, $z_{,}$ remain the same whichever element dm be taken, if we call M the whole mass of the system, we find

$$\int \frac{d^2x_{,}}{dt^2}.\,dm = \text{M}.\,\frac{d^2x_{,}}{dt^2},$$

$$\int \frac{d^2y_{,}}{dt^2}\,dm = \text{M}.\,\frac{d^2y_{,}}{dt^2}, \quad \int \frac{d^2z_{,}}{dt^2}\,dm = \text{M}.\,\frac{d^2z_{,}}{dt^2}.$$

Again, by the nature of the centre of inertia (201) since it is now the origin,

$$\int x'dm = 0, \quad \int y'dm = 0, \quad \int z'dm = 0;$$

where we find by differentiation,

$$\int \frac{d^2x'}{dt^2}.\,dm = 0, \quad \int \frac{d^2y'}{dt^2}.\,dm = 0, \quad \int \frac{d^2z'}{dt^2}.\,dm = 0.$$

The three equations obtained by the transformation become therefore

$$\text{M}.\,\frac{d^2x_{,}}{dt^2} = \int \text{X}\,dm,$$

$$\text{M}.\,\frac{d^2y_{,}}{dt^2} = \int \text{Y}\,dm, \quad \text{M}.\,\frac{d^2z_{,}}{dt^2} = \int \text{Z}\,dm.$$

By means of these equations the motion of the centre of inertia is determined and the result is the same as that obtained in art. 207; viz. that the centre of inertia moves as if the whole mass were there united, and a force equal to the resultant of all the forces were immediately applied to it.

If we make the same substitutions in the other three equations, the first of them becomes

$$\int \frac{(x_{,} + x')\,(d^2y_{,} + d^2y') - (y_{,} + y')\,(d^2x_{,} + d^2x')}{dt^2}.\,dm$$

$$= \int((x_{\prime} + x')\,Y - (y_{\prime} + y')\,X)\,dm.$$

But since $x_{\prime}$, $y_{\prime}$, $z_{\prime}$ are the same whichever element of the mass be taken,

$$\int(x_{\prime}\,d^2y_{\prime} - y_{\prime}\,d^2x_{\prime})\,dm = m\,(x_{\prime}\,d^2y_{\prime} - y_{\prime}\,d^2x_{\prime}),$$

$$\text{and } \int(x_{\prime}\,Y - y_{\prime}\,X)\,dm = x_{\prime}\int Y dm - y_{\prime}\int X dm,$$

and also

$$\int(x'\,d^2y_{\prime} - y'\,d^2x_{\prime} + x_{\prime}\,d^2y' - y_{\prime}\,d^2x')\,dm$$

$$= d^2y_{\prime}\int x'\,dm - d^2x_{\prime}\int y'\,dm + x_{\prime}\int d^2y'\,dm - y_{\prime}\int d^2x'\,dm.$$

But by the property of the centre of inertia as above,

$$\int x'\,dm = 0, \qquad \int y'\,dm = 0,$$

and therefore also

$$\int d^2x'\,dm = 0, \text{ and } \int d^2y'\,dm = 0.$$

Hence every term of the second member of the last equation becomes = 0, and there remains in the equation obtained by the transformation,

$$\int\left(\frac{x'\,d^2y' - y'\,d^2x'}{dt^2}\right)dm = \int(x'\,Y - y'\,X)\,dm.$$

Similarly the other two equations present

$$\int\left(\frac{z'\,d^2y' - y'\,d^2z'}{dt^2}\right)dm = \int(z'\,X - x'\,Z)\,dm,$$

$$\int\left(\frac{y'\,d^2z' - z'\,d^2y'}{dt^2}\right)dm = \int(y'\,Z - z'\,Y)\,dm.$$

These six equations contain the whole theory of the motion of a rigid system. The first three determine the motion of the centre of inertia, the last three the motion of the system with reference to that point as origin. We have therefore a double motion presented to us; a motion of *translation* with reference to the centre of inertia; and a motion of *rotation* round that point, as if it were fixed. If we then can ascertain at any instant the position of this centre, which may be done by investigating its motion as a separate particle, and then can ascertain the position of the system with regard to this centre, the position of the whole system is completely determined.

218. It is our object therefore to discuss the last three equations, which alone affect the position of the system, and to put them into such a form as may enable us to determine it more readily. But since this rotatory motion takes place round the centre of inertia as if it were a fixed point, we will include the investigation in the more general case of the motion of a rigid system round any fixed point; and we can easily deduce the motion when that fixed point is the centre of inertia.

CHAPTER VIII.

ON THE MOTION OF A RIGID SYSTEM WITH A FIXED POINT IN IT.

219. *Prop.* *To determine the motion of any rigid system round a fixed point.*

Let A be this fixed point (fig. 45.),

Ax, Ay, Az,	three *fixed* rectangular axes taken *arbitrarily*,
Ax', Ay', Az',	three other rectangular axes, *fixed with regard to the system*, but moving with it round A;
x, y, z,	coordinates of the element dm referred to the first set of axes,
x', y', z',	coordinates of the same dm for the moving axes.

By the principles of the transformation of coordinates[a], we have

---

[a] For the full development of the principles of the transformation of coordinates in space, the reader is referred to the works on the subject, as Hymer's Analytical Geometry, sect. 4, or Hamilton's, part ii. sect. 1. chap. 11; but it may be convenient to exhibit briefly the formulæ which are necessary for the present chapter.

Since
$$x = ax' + by' + cz',$$
$$y = a'x' + b'y' + c'z',$$
$$z = a''x' + b''y' + c''z',$$

where a, b, c are the cosines of the angles which the axis of x makes with the axes of x', y', z' respectively, &c.

$$x = ax' + by' + cz',$$
$$y = a'x' + b'y' + c'z',$$
$$z = a''x + b''y' + c''z',$$

in which expressions a, b, c represent the cosines of the angles which the axis of x makes with the axes of x', y', z'; a', b', c'

The coefficients $a, b, c;\ a', b', c';\ a'', b'', c''$, are connected by the equations,

$$a^2 + a'^2 + a''^2 = 1, \qquad ab + a'b' + a''b'' = 0,$$
$$b^2 + b'^2 + b''^2 = 1, \qquad ac + a'c' + a''c'' = 0,$$
$$c^2 + c'^2 + c''^2 = 1, \qquad bc + b'c' + b''c'' = 0.$$

If we refer reciprocally the coordinates x, y, z to the axes of x', y', z', we have

$$x' = ax + a'y + a''z,$$
$$y' = bx + b'y + b''z,$$
$$z' = cx + c'y + c''z.$$

Hence also we obtain the equations

$$a^2 + b^2 + c^2 = 1, \qquad aa' + bb' + cc' = 0,$$
$$a'^2 + b'^2 + c'^2 = 1, \qquad aa'' + bb'' + cc'' = 0,$$
$$a''^2 + b''^2 + c''^2 = 1, \qquad a'a'' + b'b'' + c'c'' = 0.$$

These last six equations are equivalent to the first six, and by either set the nine variables are connected as equations of condition, so that three only may be considered as arbitrary.

If again, by the usual process of elimination, we obtain from the first equations values of x', y', z' in terms of x, y, z and the coefficients, and compare the results with the values given in the second set, we have new relations between them, viz.

$$a = b'c'' - b''c', \qquad a' = b''c - bc'', \qquad a'' = bc' - b'c,$$
$$b = a''c' - a'c'', \qquad b' = ac'' - a''c, \qquad b'' = a'c - ac',$$
$$c = a'b'' - a''b', \qquad c' = a''b - ab'', \qquad c'' = ab' - a'b.$$

the cosines of the angles which the axis of y makes with the same axes, &c., and therefore these quantities are connected by the usual equations of condition.

Also, since only three of these variables may be considered as independent quantities, it is both natural and convenient to obtain expressions for them in terms of three other independent quantities.

Thus, if AT (fig. 46.) be the intersection of the plane of x', y' with that of x, y, or the trace of the plane of x, y, on that of x', y',

$$\text{let } \angle \text{TA}x = \psi, \qquad \angle \text{TA}x' = \phi,$$

and let the inclination of the plane of x', y' to that of x, y, or, which is the same thing, the $\angle$ $z\text{A}z' = \theta$, since the axes Az, Az' are perpendicular to those two planes.

Then suppose a sphere to be described with radius $= 1$, and A as a centre, we have, on the surface of this sphere, a triangle formed by the axes intercepted by the angles TAx, TAx', and xAx', and hence, since by the formula of Spherical Trigonometry, α, β, γ being the sides of a triangle, and A the angle opposite to α,

$$\cos.\,\alpha = \sin.\,\beta.\,\sin.\,\gamma.\,\cos.\,\text{A} + \cos.\,\beta.\,\cos.\,\gamma,$$

we obtain in this case

$$\cos.\,x\text{A}x' = a = \sin.\,\psi.\,\sin.\,\phi.\,\cos.\,\theta. + \cos.\,\psi.\,\cos.\,\phi.$$

Again, in the triangle formed by the arcs intercepted by TAx, TAy', and y'Ax, observing that TA$y' = \frac{\pi}{2} + \phi$, we obtain

$$\cos.\,x\text{A}y' = b = \sin.\,\psi.\,\cos.\,\phi.\,\cos.\,\theta. - \cos.\,\psi.\,\sin.\,\phi.$$

Again, in the triangle formed by the arcs intercepted by TAx, TAz', and z'Ax, since TA$x' = \frac{\pi}{2}$, we obtain

$$\cos.\,x\text{A}z' = c = \sin.\,\psi.\,\sin.\,\theta.$$

Similarly we can find

$$a' = \cos.\,\psi.\,\sin.\,\phi.\,\cos.\,\theta. - \sin.\,\psi.\,\cos.\,\phi,$$

$$b' = \cos.\,\psi.\,\cos.\,\phi.\,\cos.\,\theta. + \sin.\,\psi.\,\sin.\,\phi,$$

$$c' = \cos.\,\psi.\,\sin.\,\theta,$$

Now since the axes Ax', Ay', Az' remain fixed with regard to the system during the whole motion, it follows that the coordinates x', y', z' are independent of the time t, while the quantities a, b, c, &c. vary with it. Differentiating, therefore, on this supposition, we have

$$\frac{dx}{dt} = x'\frac{da}{dt} + y'\frac{db}{dt} + z'\frac{dc}{dt},$$

$$\frac{dy}{dt} = x'\frac{da'}{dt} + y'\frac{db'}{dt} + z'\frac{dc'}{dt},$$

$$\frac{dz}{dt} = x'\frac{da''}{dt} + y'\frac{db''}{dt} + z'\frac{dc''}{dt}.$$

The values of $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are the velocities at any instant parallel to the axes Ax, Ay, Az, of a point whose coordinates are x', y', z' referred to the other axes.

Now if the system has any rotatory motion round the fixed point, there may be more points than one whose velocity at any instant may be nothing, and these points may be determined by making the above expressions equal to nothing. In order to determine these points, multiply the first of the equations by c, the second by c', and the third by c'', and we find by addition

$$a'' = -\sin.\phi.\sin.\theta,$$

$$b'' = -\cos.\phi.\sin.\theta,$$

$$c'' = \cos.\theta.$$

By substituting these nine values in the original formulæ, the values x, y, z are obtained in terms of the three independent variables ψ, ϕ, θ.

$$\left.\begin{aligned} &\frac{cda + c'da' + c''da''}{dt} . \; x' \\ &\frac{cdb + c'db' + c''db''}{dt} . \; y' \\ &\frac{cdc + c'dc' + c''dc''}{dt} . \; z' \end{aligned}\right\} = 0.$$

But since

$c^2 + c'^2 + c''^2 = 1$, we have $cdc + c'dc' + c''dc'' = 0$,

and putting

$$cdb + c'db' + c''db'' = pdt,$$

$$cda + c'da' + c''da'' = -qdt,$$

the expression becomes

$$py' - qx' = 0.$$

Again, multiplying the first equation by b, the second by b', and the third by b'', we obtain a similar expression, which may be similarly reduced, since by the nature of the coefficients,

$$b^2 + b'^2 + b'^2 = 1,$$

and $$bc + b'c' + b''c'' = 0;$$

whence $$bdb + b'db' + b''db'' = 0,$$

and

$$bdc + b'dc' + b''dc'' = -cdb - c'db' - c''db'' = -pdt.$$

Putting therefore $bda + b'da' + b''da'' = rdt$, we have

$$rx' - pz' = 0.$$

Similarly, multiplying the first by a, the second by a', and the third by a'', and reducing, we have

$$qz' - ry' = 0.$$

These three equations thus obtained, viz.

$$py' - qx' = 0,$$

$$rx' - pz' = 0,$$

$$qz' - ry' = 0,$$

two of which involve the other, are evidently the equations to a straight line passing through the origin of coordinates, or the fixed point A.

Hence we conclude, that all the points whose velocity at any instant is nothing, are ranged along a straight line passing through the fixed point. This straight line may be supposed to remain unmoved during an indefinitely small time, and the system therefore turns round this line for that time, as if it were a fixed axis: and we may consequently represent the motion as taking place at every instant round an axis, which during that instant remains unmoved. The position of this axis generally changes at every instant, and it is for this reason called the *Instantaneous axis of rotation.*

220. PROP. *To determine the position of the instantaneous axis of rotation.*

Since the three equations deduced in the last proposition are the equations of the projections of this line on the planes of x', y', z', we have by the principles of projections if A I (fig. 45) represent the line,

$$\cos. \mathrm{I A} x' = \frac{p}{\sqrt{p^2 + q^2 + r^2}},$$

$$\cos.\,\mathrm{I}\,\mathrm{A}\,y' = \frac{q}{\sqrt{p^2 + q^2 + r^2}},$$

$$\cos.\,\mathrm{I}\,\mathrm{A}\,z' = \frac{r}{\sqrt{p^2 + q^2 + r^2}}.$$

If then the three quantities p, q, r, are known, we can ascertain the position of the instantaneous axis with reference to three moving ones, $\mathrm{A}\,x'$, $\mathrm{A}y'$, $\mathrm{A}z'$.

If these quantities are constant, the axis will remain fixed within the system, and the system will rotate, as it would round an axis fixed in space.

221. Prop. *To determine the motion of rotation round the fixed point.*

If we integrate the three last of the general equations (216), and represent the sums of the moments of the forces acting on the several parts of the system with reference to the three axes respectively, by M, M′, M″, so that

$$\mathrm{M} = \int (y\mathrm{Z} - z\mathrm{Y})\, dm,$$

$$\mathrm{M}' = \int (z\mathrm{X} - x\mathrm{Z})\, dm, \qquad \mathrm{M}'' = \int (x\mathrm{Y} - y\mathrm{X})\, dm,$$

we have the three equations

$$\int \left(\frac{y\,dz - z\,dy}{dt}\right).\, dm = \int \mathrm{M}\, dt,$$

$$\int \left(\frac{z\,dx - x\,dz}{dt}\right).\, dm = \int \mathrm{M}'\, dt,$$

$$\int \left(\frac{x\,dy - y\,dx}{dt}\right)\, dm = \int \mathrm{M}''\, dt.$$

Then by substituting for x, y, z, and their differential coefficients $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, their values given in art. 219, observing that, a, b, c, &c. are variable, but x', y', z', constant for the same part of the mass, the first of the three equations becomes,

$$\int \left\{ \begin{array}{l} a'.\frac{da''}{dt}.x'^2 + a'.\frac{db''}{dt}.x'y' + a'.\frac{dc''}{dt}.x'z', \\ + b'.\frac{da''}{dt}.x'y' + b'.\frac{db''}{dt}.y'^2 + b'.\frac{dc''}{dt}.y'z', \\ + c'.\frac{da''}{dt}.x'z' + c'.\frac{db''}{dt}.y'z' + c'.\frac{dc''}{dt}.z^2, \\ - a''.\frac{da'}{dt}.x'^2 - a''.\frac{db'}{dt}.x'y' - a''.\frac{dc'}{dt}.x'z', \\ - b''.\frac{da'}{dt}.x'y' - b''.\frac{db'}{dt}.y'^2 - b''.\frac{dc'}{dt}.y'z', \\ - c''.\frac{da'}{dt}.x'z' - c''.\frac{db'}{dt}.y'z' - c''\frac{dc'}{dt}.z'^2, \end{array} \right\} dm = \int \mathrm{M}dt.$$

In this equation we observe that the cooefficient of x'^2 is $\frac{a'da'' - a''da'}{dt}$. Hence since by the property of the coefficients

$$a' = b''c - bc'', \text{ and } a'' = bc' - b'c,$$

we obtain by substitution,

$$\begin{aligned} a'da'' - a''da' &= b''cda'' - bc''da'' - bc'da' + b'cda' \\ &= c\,(bda + b'da' + b''da'') \\ &\quad - b\,(cda + c'da' + c''da') \\ &= crdt + bqdt, \end{aligned}$$

by introducing q and r, as in art. 219,

Hence the coefficient is

$$cr + bq,$$

Similar the coefficient of y'^2, which is $\dfrac{b'db'' - b''db'}{dt}$, will become

$$ap + cr;$$

and also the coefficient of x'^2, which is $\dfrac{c'dc'' - c''dc'}{dt}$, will become

$$bq + ap.$$

Collecting therefore these terms together and supposing at present that the coefficients are so assumed as to make the other terms, viz. those which involve $(x'y', x'z', \text{\&}\ y'z')$ disappear, we obtain,

$$\int \left\{ \begin{array}{l} \quad (cr + bq)\, x'^2 \\ + (ap + cr)\, y'^2 \\ + (bq + ap)\, z'^2 \end{array} \right\} dm = \int \mathrm{M} dt,$$

$$\therefore \int \left\{ cr\,(x'^2 + y'^2) + bq\,(x'^2 + z'^2) + ap\,(y'^2 + z'^2) \right\} dm = \int \mathrm{M} dt.$$

Again, assuming

$$\mathrm{A} = \int (y'^2 + z'^2)\, dm, \quad \mathrm{B} = \int (x'^2 + z'^2)\, dm, \quad \mathrm{C} = \int (x'^2 + y'^2)\, dm,$$

The equation becomes

$$\mathrm{A}ap + \mathrm{B}bq + \mathrm{C}cr = \int \mathrm{M}dt.$$

By similar substitutions the other two equations become

$$\mathrm{A}a'p + \mathrm{B}b'q + \mathrm{C}c'r = \int \mathrm{M}'dt$$

$$\mathrm{A}a''p + \mathrm{B}b''q + \mathrm{C}c''r = \int \mathrm{M}''dt.$$

In order to separate the variables a, b, c, &c. from p, q, r, we must differentiate these three equations, and if we then multiply the first by a, the second by a', and the third by a'', we obtain by addition

$$\mathrm{A}.\frac{dp}{dt} + (\mathrm{C} - \mathrm{B}).\, qr = a\mathrm{M} + a'\mathrm{M}' + a''\mathrm{M}''.$$

Similarly, by multiplying the first differential equation by b, the second by b', and the third by b'', we obtain,

$$\mathrm{B}.\frac{dq}{dt} + (\mathrm{A} - \mathrm{C})\, rp = b\mathrm{M} + b'\mathrm{M}' + b''\mathrm{M}'',$$

and similarly,

$$\mathrm{C}.\frac{dr}{dt} + (\mathrm{B} - \mathrm{A})\, pq = c\mathrm{M} + c'\mathrm{M}' + c''\mathrm{M}'.$$

We have here then three equations arising from the transformation of the last three general equations, and these are sufficient to determine the motion of rotation of the system. By integration we shall find the values of p, q, r, and combining these three expressions thus obtained with the six equations which necessarily connect the nine variables a, b, c, a', b', c', a'', b'', c'', we shall obtain the values of the variables by a new integration. The values of these variables will enable us to determine at any instant the position of the moving axes $\mathrm{O}x'$, $\mathrm{O}y'$, $\mathrm{O}z'$, and as their position is fixed with regard to the system, the position of the system is thus determined.

222. The three equations just obtained may be more conveniently expressed by transforming the moments of the forces with reference to the axes Ox, Oy, Oz, which we assumed to be M, M′, M″, to the new axes. For if N, N′, N″, are the moments of the forces with reference to the new axes, we have at once, by the principles of transformation,

$$N = aM + a'M' + a''M'',$$
$$N' = bM + b'M' + b''M'',$$
$$N'' = cM + c'M' + c''M''.$$

Whence the three equations become

$$A\,dp + (C - B).\,qr\,dt = N\,dt$$
$$B\,dq + (A - C)\,rp\,dt = N'dt$$
$$C\,dr + (B - A)\,pq\,dt = N''dt.$$

223. In the demonstration of the proposition it was supposed that the coefficients a, b, c, &c. might be so assumed as to satisfy the equations

$$\int x'y'\,dm = 0, \quad \int x'z'\,dm = 0, \quad \int y'z'\,dm = 0.$$

That such an assumption may be made, can be demonstrated from the nature of the coefficients: and the axes which are obtained by this assumption possess some remarkable properties, and are called the *principal axes.*

224. Prop. *To determine the principal axes of any rigid system.*

Here it is requisite to shew that there may be any principal axes, or rather that axes may be taken so as to fulfil the conditions. Now the principles of the transformation (see note, art. 219) enable us to determine the variables a, b, c, &c. in terms of three other independent quantities, which will determine the position of any three rectangular axes: and if the inclination of the planes x, y, and x', y', be called θ; and the angles which the intersection of these two planes makes with the axes of x and x' respectively be called ψ and ϕ, we obtain

$$\begin{aligned} x' &= x\,(\sin.\,\psi.\,\sin.\,\phi.\,\cos.\,\theta + \cos.\,\psi.\,\cos.\,\phi) \\ &\quad + y\,(\cos.\,\psi.\,\sin.\,\phi.\,\cos.\,\theta - \sin.\,\psi.\,\sin.\,\phi) - z\,\sin.\,\phi.\,\sin.\,\theta, \\ y' &= x\,(\sin.\,\psi.\,\cos.\,\phi.\,\cos.\,\theta - \cos.\,\psi.\,\sin.\,\phi) \\ &\quad\quad y\,(\cos.\,\psi.\,\cos.\,\phi.\,\cos.\,\theta + \sin.\,\psi.\,\sin.\,\phi) - z\,\cos.\,\phi.\,\sin.\,\theta, \\ z' &= x\,\sin.\,\psi.\,\sin.\,\theta + y\,\cos.\,\psi.\,\sin.\,\theta + z\,\cos.\,\theta; \end{aligned}$$

or putting, for the sake of abbreviation,

$$\begin{aligned} \mathrm{X} &= x\,\cos.\,\psi - y\,\sin.\,\psi, \\ \mathrm{Y} &= x\,\sin.\,\psi\,\cos.\,\theta + y\,\cos.\,\psi.\,\cos.\,\theta - z\,\sin.\,\theta, \end{aligned}$$

we have

$$\begin{aligned} x' &= \mathrm{Y}\,\sin.\,\phi + \mathrm{X}\,\cos.\,\phi, \\ y' &= \mathrm{Y}\,\cos.\,\phi - \mathrm{X}\,\sin.\,\phi, \\ z' &= x\,\sin.\,\psi.\,\sin.\,\theta + y\,\cos.\,\psi.\,\sin.\,\theta + z\,\cos.\,\theta. \end{aligned}$$

If we then substitute these values in the three expressions,

$$\int x'y'dm = 0, \qquad \int x'z'dm = 0, \qquad \int y'z'dm = 0,$$

we find from the first,

$$\left.\begin{aligned} &\sin.\,\phi.\,\cos.\,\phi.\int(\mathrm{Y}^2 - \mathrm{X}^2)\,dm \\ &+ (\cos.^2\,\phi. - \sin.^2\,\phi.)\int \mathrm{X}\,\mathrm{Y}\,dm \end{aligned}\right\} = 0,$$

or $\sin.\,2\,\phi.\int(\mathrm{Y}^2 - \mathrm{X}^2)\,dm + 2\cos.\,2\,\phi.\int \mathrm{X}\,\mathrm{Y}\,dm = 0.$ *(m)*

So from the others,

$$\cos.\,\phi.\int \mathrm{Y}\,z'\,dm \;-\; \sin.\,\phi.\int \mathrm{X}\,z'\,dm \;=\; 0, \qquad (n)$$

$$\sin.\,\phi.\int \mathrm{Y}\,z'\,dm \;+\; \cos.\,\phi.\int \mathrm{X}\,z'\,dm \;=\; 0. \qquad (p)$$

From these last two equations (n and p) we readily obtain,

$$\int \mathrm{Y}\,z'\,dm = 0, \qquad \int \mathrm{X}\,z'\,dm = 0.$$

Now these two equations do not involve the angle φ, and hence we may from them find values of ψ and θ which may satisfy the required conditions: and if these angles are known, the first equation will enable us to determine the tan. 2φ, and therefore the angle φ.

Thus by making the proper substitutions,

$$\mathrm{Y}z' = \sin.\theta.\cos.\theta\,(x^2\sin.^2\psi + 2xy\,\sin.\psi.\cos.\psi + y^2\cos.^2\psi - z^2)$$
$$+\;(\cos.^2\theta - \sin.^2\theta)\;(xz\,\sin.\psi + yz\,\cos.\psi),$$

$$\mathrm{X}\,z' = \sin.\,\theta\,\left\{xy\,(\cos.^2\psi - \sin.^2\psi) + (x^2 - y^2)\sin.\psi.\cos.\psi\right\}$$
$$+\;\cos.\,\theta\,(xz\,\cos.\psi - yz\,\sin.\psi).$$

If we then assume

$$\int x^2dm = \mathrm{A}', \qquad \int y^2dm = \mathrm{B}', \qquad \int z^2dm = \mathrm{C}',$$
$$\int xydm = \mathrm{E}', \qquad \int xzdm = \mathrm{F}', \qquad \int yzdm = \mathrm{H}',$$

and observe that these quantities A, B, C, E, F, H, are constants which depend on the nature of the body and the direction of the axes, which we have chosen arbitrarily, we obtain

$$\left.\begin{array}{l}\sin.\,2\theta\,(\mathrm{A}'\,\sin.^2\psi \;+\; 2\mathrm{E}'\,\sin.\psi\,\cos.\psi \;+\; \mathrm{B}'\,\cos.^2\psi \\ \qquad -\;\mathrm{C}') + 2\cos.\,2\theta\,(\mathrm{F}'\,\sin.\psi + \mathrm{H}'\,\cos.\psi)\end{array}\right\} = 0,$$

$$\left.\begin{array}{l}\sin.\theta\left\{\mathrm{E}'\,(\cos.^2\psi - \sin.^2\psi) + (\mathrm{A}' - \mathrm{B}')\sin.\psi.\cos.\psi\right\} \\ \qquad +\;\cos.\,\theta\,(\mathrm{F}'\,\cos.\psi - \mathrm{H}'\,\sin.\psi)\end{array}\right\} = 0.$$

From the first of this the value of tan. 2θ may be obtained, and from the second the value of tan. θ, and θ may be eliminated by means of the trigonometrical formula,

$$\tan. 2\theta = \frac{2 \tan. \theta}{1 - \tan.^2 \theta}.$$

In order to effect this more conveniently assume tan. $\psi = u$.

$$\therefore \sin. \psi = \frac{u}{\sqrt{1 + u^2}}, \cos. \psi = \frac{1}{\sqrt{1 + u^2}},$$

$$\therefore \tan. 2\theta = -\frac{2(F'u + H')\sqrt{1 + u^2}}{A'u^2 + 2E'u + B' - C'(1 + u^2)},$$

$$\tan. \theta = -\frac{(F' - H'u)\sqrt{1 + u^2}}{E'(1 - u^2) + (A' - B')u}.$$

Hence making the proper substitutions and reducing we obtain,

$$\{E'(1 - u^2) + (A' - B')u\} \{B'F' - F'C' - E'H' \\ + (C'H' - A'H' + E'F')u\} \\ + (F'u + H')(F' - H'u)^2 = 0.$$

We have here an equation of the third degree, which must have at least one real value of u, and therefore must give us a real value for the angle ψ: and if we substitute this value in either of these last two equations, we shall find a real value for θ. Hence since we can obtain from equation (m) a value for the tan. 2φ. and therefore for φ, it follows that φ, ψ, and θ, can always be assumed so as to satisfy the conditions, and hence there may be at least one system of *principal axes*, which cut one another in the point A, and which may be determined by assuming the angles φ, ψ, θ, so as to fulfil the equations,

$$\int x'y'dm = 0, \int x'z'dm = 0, \qquad \int y'z'dm = 0.$$

225. The three roots of the cubic equation obtained in the last article are necessarily real, but we must not therefore conclude that there exists in each system more than one system of principal axes. In fact, these three roots represent the tangents of the angles contained by the axis of x, and each of the three lines in which the planes of the coordinates x', y', z', cut the plane of x, y; for there is no reason why we should consider one of these angles as given by the equation rather than the other, and the equations of condition are equally satisfied, when we interchange the axes of x', y', z'. Hence, in general, there exists only one system of principal axes; and in order that there might be more, the equation ought to be of a higher degree, and should present three times as many real roots as there should be systems.

But yet if the equations which determine the values of φ, ψ, θ, become identical, we then have an infinite number of systems of principal axes.

226. *Def.* The sum of the elements of any system multiplied respectively by the squares of their distances from any axis, is called the *moment of inertia of the system with reference to that axis.*

Thus the quantities which in art. 221, were designated by A, B, C, represent the moments of inertia of the system referred respectively to the axes of x', y', z', respectively.

The value of the moment of inertia varies necessarily with the position of the axis to which it is referred, but when the moments of inertia with reference to the principal axes are given, it is easy to determine the moment with reference to any other axis whatever.

227. **Prop.** *To determine the moment of inertia of any system with reference to any axis having the same origin as the principal axes, when the moments with reference to the principal axes are given.*

Let x', y', z', be the coordinates of any element of the system dm referred to the principal axes,

x, y, z, the coordinates of the same element referred to any other axes with the same origin.

And let $C' = \int (x^2 + y^2)\, dm$, the moment of the system with reference to the axis of z.

If we substitute the values of x, y, obtained by the principles of transformation (note 219) we have since $\int x'y'\, dm = 0$, &c.

$$C' = (1 - a''^2) \int x'^2 dm + (1 - b''^2) \int y'^2 dm + (1 - c''^2) \int z'^2 dm.$$

And since $a''^2 + b''^2 + c''^2 = 1$, this equation may be reduced to

$$C' = a''^2 . A + b''^2 . B + c''^2 . C.$$

The three quantities a'', b'', c'', are the cosines of the angles which the axis of z makes with each of the three principal axes; hence, the moment of inertia of a system with reference to any axis passing through a given point is generally equal to the sums of the moments with reference to the principal axes which cross at that point, multiplied respectively by the squares of the cosines of the angles which the axis in question makes with the principal axes.

228. **Prop.** *To determine with respect to which of the axes drawn through the same point, the moment of inertia of any system is a maximum or minimum.*

Let A be the greatest, and C the least of the three moments, A, B, C referred to the three principal axes.

Then substituting for a''^2 its value $1 - b''^2 - c''^2$ in the equation for C′ found in the last art. we have,

$$C' = A - b''^2 (A - B) - c''^2 (A - C).$$

By hypothesis $A - B$ and $A - C$ are positive; hence C′ is always less than A, whatever be the values of b'' and c''.

Again, substituting in the same equation for c''^2 its value $1 - a''^2 - b''^2$, we have

$$C' = C + a''^2 (A - C) + b''^2 (B - C).$$

Hence C′ is always greater than C.

We conclude, therefore, that the *principal axes* exhibit generally the *greatest* and the *least* moments of all axes through the same point.

229. *Cor.* 1. If $A = B$, we have

$$C' = (1 - c''^2) A + c''^2 C,$$

and as this value only depends on c'', it follows that the moment of inertia is in this case the same for all axes making the same angle with the axis of z'. The moments of inertia, therefore, referred to all the axes in the plane of x', y', which make a right angle with the axis of z', are equal to one another; but, in this case, every system consisting of the axis of z', and two other axes at right angles to each other and to it, are principal axes: or for any such system,

$$\int xy\,dm = 0, \qquad \int xz\,dm = 0, \qquad \int yz\,dm = 0.$$

For if we transform these expressions to the axes of x', y' z', we obtain, observing that $\int x'y'\,dm = 0$, &c.

$$aa'\int x'^2 dm + bb'\int y'^2 dm + cc'\int z'^2 dm = 0,$$

$$aa''\int x'^2dm + bb''\int y'^2dm + cc''\int z'^2dm = 0,$$

$$a'a''\int x'^2dm + b'b''\int y'^2dm + c'c''\int z'^2dm = 0,$$

But by hypothesis $A = B$

$$\therefore \int y'^2dm = \int x'^2dm,$$

and also $$aa' + bb' + cc' = 0, \text{ \&c.}$$

Hence the three equations give us,

$$cc'\,(\int x'^2dm - \int z'^2dm) = 0,$$

$$cc''\,(\int x'^2dm - \int z'^2dm) = 0,$$

$$c'c''\,(\int x'^2dm - \int z'^2dm) = 0.$$

These equations are satisfied by supposing $c = 0$, $c' = 0$, and $c'' = 1$; whence it follows that all the axes in the plane of x', y', are principal axes, and the system has an infinite number of principal axes, which have the axis of z' common to them all.

230. *Cor.* 2. If at the same time $A = B = C$, we have

$$C' = A.$$

The moments of inertia, therefore, are all equal, and all the axes are principal axes: for the equations obtained by the transformation in the last article are satisfied independently of the values of a, b, c, &c.

231. Prop. *To determine the moment of inertia of any system with reference to any axis, when the moment with reference to an axis parallel to it and passing through the centre of inertia is given.*

Let X, Y, Z be the coordinates of the centre of inertia,

$x_{,}$, $y_{,}$, $z_{,}$ the coordinates of the element dm referred to that point, so that $x = x_{,} + \mathrm{X}$, $y = y_{,} + \mathrm{Y}$, $z = z_{,} + \mathrm{Z}$.

Then, by substitution, we obtain

$$\begin{aligned} \mathrm{C}' &= \int \{ (x_{,} + \mathrm{X})^2 + (y_{,} + \mathrm{Y})^2 \} dm, \\ &= \int (x_{,}^2 + y_{,}^2) dm + 2\mathrm{X} \int x_{,} dm + 2\mathrm{Y} \int y_{,} dm \\ &\qquad + (\mathrm{X}^2 + \mathrm{Y}^2) \int dm. \end{aligned}$$

But by the property of the centre of inertia (201)

$$\int x_{,} dm = 0, \qquad \text{and} \int y_{,} dm = 0.$$

If therefore we call M the mass of the whole system, and r the distance of the centre of inertia from the axis of z, we have

$$\mathrm{C}' = \int (x_{,}^2 + y_{,}^2) \, dm + r^2 \mathrm{M}.$$

Hence the moment of inertia of any system with reference to a given axis, is equal to the moment with reference to a parallel axis through the centre of inertia *plus* the moment of the whole mass collected in its centre of inertia with reference to the given axis.

This property enables us to find more easily the moment of inertia of any system, by first finding the moment with reference to an axis through the centre of inertia.

232. *Cor.* Hence it is evident, that the least of all the moments of any system is referred to one of the principal axes which cross at the centre of inertia.

CHAPTER IX.

ON THE MOTION OF A RIGID SYSTEM ROUND A FIXED AXIS.

233. If there are two fixed points in the system, the motion can only take place round the line joining these two points, which therefore becomes a fixed axis (50).

Prop. To determine the motion of a system round a fixed axis.

The investigation of the motion is in this case more simple and it is more convenient to refer to the general equations deduced in art. 206 rather than to the equations of rotation in art. 221. We can here assume the fixed axis of rotation for one of the axes of coordinates, as for instance, for the axis of z, and the plane of x, y, is consequently perpendicular to the axis of rotation. Again, this plane of x, y, may be assumed to pass through the centre of inertia of the system; and the first of the three general equations will determine the motion, viz.

$$\int\left(\frac{x d^2 y - y d^2 x}{dt^2}\right) dm = \int(x\mathrm{Y} - y\mathrm{X})\, dm.$$

If we suppose a plane to pass through the axis of rotation and the centre of inertia of the system, the position of the *trace* of this plane on the plane of x, y, will determine at any instant the position of the whole system. Let us therefore assume our new axes of x', y', z', such that the axis of x' may coincide with this trace, and the axis of z' with that of z or the fixed axis; and let θ be the angle which this trace or the new axis of x' makes with the axis of x, then we have

$$x = x' \cos. \theta + y' \sin. \theta, \qquad y = - x' \sin. \theta + y' \cos. \theta.$$

If we differentiate these values, observing that x', and y' are constant for the same element dm, but the angle θ variable, and if we substitute in the equation, we obtain

$$\frac{d^2 \theta}{dt^2} \int (x'^2 + y'^2)\, dm = - \int (x\text{Y} - y\text{X})\, dm.$$

The expression $\int (x^2 + y'^2)\, dm$ is the moment of inertia of the system with reference to the axis of z, and if we call this moment A, the equation becomes,

$$\text{A}.\frac{d^2 \theta}{dt^2} = - \int (x\text{Y} - y\text{X})\, dm.$$

Since the parts of the system are supposed to be connected in an invariable manner, every particle must revolve round the axis with the same angular velocity; and since θ is the angle which the revolving *trace* makes with the axis of x or with a fixed line, $\frac{d\theta}{dt}$ will measure the rate of increase of this angle, or will measure the angular velocity of the trace, and therefore of the whole system.

234. *Cor.* 1. From the above expression then we can obtain by integration the angular velocity of the system, if we know the forces which act upon it.

Thus if the only force acting on the system be an *uniform* force acting parallel to the axis of x, so that $\text{Y} = 0$ and $\text{X} = f$ we then have

$$\begin{aligned} \int (x\text{Y} - y\text{X})\, dm &= - \int y\, \text{X} dm \\ &= f \sin. \theta \int x' dm - f \cos. \theta \int y' dm, \end{aligned}$$

since θ is constant for any one position of the system.

But since also the axis x' passes through the centre of inertia of the system $\int y' dm = 0$ and $\int x' dm = M x_{,}$ (201), if M is the whole mass of the system. Hence

$$A \frac{d^2 \theta}{d t^2} = - M x_{,} f \sin. \theta.$$

Multiplying this by $2 d \theta$ and integrating, we have,

$$\frac{d \theta^2}{d t^2} = \frac{2 M x_{,} f \cos. \theta}{A} + C,$$

C being an arbitrary constant.

235. The importance of this case of motion round an axis may perhaps require that an investigation should be given independent of the general equations; and therefore the following is deduced from the general principle of d'Alembert (204).

Suppose the system to be acted on by any forces which cause it to revolve round the fixed axis.

Let F be any accelerating force acting on any element dm and making an angle δ with the tangent to the circle in which the element is compelled to move. And if F be resolved into three directions, one parallel to the fixed axis and therefore ineffectual, the second in the direction of the radius of the circle, and therefore destroyed by the axis, and the third in the direction of the tangent; then this last, which will be represented by $F \cos. \delta$, will be the effective accelerating force.

If then ω is the angular velocity of each particle at the end of any time t,
r the distance of the element dm from the axis,

$r \omega$ will be the velocity of dm at the end of t: but the accelerating force would tend to impress upon the element during any instant dt a velocity represented by $F \cos. \delta. dt$; and hence at the end of any time $t + dt$, we have an *impressed* velocity on dm represented by $r\omega + \mathrm{F} \cos. \delta. dt$.

But since the element is connected with the rest of the system it can only receive during the time dt a velocity $r d\omega$, and therefore the *actual* velocity will be represented by $r\omega + r d\omega$.

Multiplying therefore these expressions by dm, we obtain the values of the impressed and actual motions, and again multiplying by the distance of dm from the axis we obtain the moments of these forces which ought to be in equilibrio round that axis. Hence by the principle (204) we obtain for the whole system,

$$\int (r^2 \omega + \mathrm{F} r \cos. \delta. dt)\, dm - \int (r^2 \omega + r^2 d\omega)\, dm = 0.$$

Whence we obtain,

$$dt \int \mathrm{F} r \cos. \delta. dm = d\omega \int r^2 dm,$$

$$\therefore \frac{d\omega}{dt} = \frac{\int \mathrm{F} r \cos. \delta. dm}{\int r^2 dm}.$$

This equation will enable us to determine the angular velocity of the system in terms of the time, when the force F and the angle δ are given for every point in the system.

236. If the system revolve uniformly round the axis, let the force which put it into motion be such as would have impressed on any particle, m, a velocity v, were the particle free.

Let θ be the angle which the direction of the force makes with a plane perpendicular to the axis, then as the motion can only take place in this plane, the velocity which F really tends to impress on m is represented by $v \cos. \theta$. The actual velocity in this case is $r\omega$. Hence if p represents the perpendicular let fall from the axis on the direction of the motion, we have by the principle, as in the last art.

$$\Sigma. m p v \cos. \theta = \Sigma. m r^2 \omega = \omega. \Sigma. m r^2.$$

If the same velocity is impressed on all the particles acted on, and if it is in the plane perpendicular to the axis, we have $\theta = 0$, and $\cos.\ \theta = 1$, and hence,

$$v.\ \Sigma.\ mp = \omega.\ \Sigma.\ mr^2.$$

And if h be the perpendicular let fall from the centre of inertia of the whole mass M, on a plane passing through the axis of rotation, and parallel to the directions of the impressed velocity, we then have $\mathrm{M}\,h = \Sigma.\ mp$, and therefore

$$\omega = \frac{\mathrm{M}\,v\,h}{\Sigma.\ mr^2}.$$

Or, if the system is continuous and the particles in contact,

$$\omega = \frac{\mathrm{M}\,v\,h}{\int r^2\,dm}.$$

If only a certain portion of the mass is acted on, as μ,

$$\omega = \frac{\mu\,v\,h}{\int r^2 dm}.$$

This expression will enable us to determine the angular velocity of the system.

237. The *pressure* upon the axis must be estimated by ascertaining what part of the forces are counteracted by the axis; but besides this pressure arising from the accelerating forces, there is another, which exists in every system in motion, arising from the inertia of the several parts. This tendency, which in a rotatory motion would make the particle endeavour to fly off from the centre of rotation, and has therefore been called a centrifugal force (197), will, if not counteracted, produce a pressure or strain upon the axis, which we now proceed to estimate.

Prop. To determine the pressure upon a fixed axis of rotation arising from the inertia of the parts of the system.

Let the axis of rotation be assumed as the axis of z; x, y, z being the coordinates of any element dm,

r, the distance of dm from the fixed axis or the radius of the circle, which it describes in the revolution,

ω, the angular velocity common to all the points.

Then, if we suppose the system to revolve uniformly, or not acted on by any accelerating forces, $r\omega$ will measure the actual velocity of dm; and hence the centrifugal force of the element will be (198) $r\omega^2$, and this force will act in the direction of the radius produced. The moving force of this element will be therefore $r\omega^2\,dm$; and as each particle will exert a similar action upon the axis, the resultant of all these forces, or the two resultants, if they cannot be compounded into one, will give the whole pressure on the axis arising from the inertia.

In order to estimate this, let the point of application of the force $r\omega^2 dm$ be transferred to the point where its direction cuts the axis; and let the force be there resolved into two, respectively parallel to the axes of x and of y, and situated in the planes of x, z, and y, z. Since, therefore, the direction of the force is inclined to the axes at angles whose cosines are $\frac{x}{r}$ and $\frac{y}{r}$, the components are

$$x\omega^2 dm, \qquad y\omega^2 dm;$$

and the resultant of all the forces parallel to the axis of x is

$$\int x\,\omega^2 dm = \omega^2 \int x\,dm = \omega^2 . \mathrm{M}x_{\prime},$$

if M represents the mass of the system, and $x_{\prime}$ the coordinate of the centre of inertia.

Similarly, the resultant of all the forces parallel to the axis of y, is

$$\int y\omega^2 dm = \omega^2 \int y\,dm = \omega^2 . My.$$

And if z', z'' are the distances of these resultants from the plane of x, y, we have, by the theory of the moments of parallel forces (27),

$$Mx_, z' = \int xz\,dm, \qquad My_, z'' = \int yz\,dm;$$

from which z' and z'' may be determined.

We thus discover the forces which acting in the planes of x, z, and y, z, endeavour to draw the axes in a direction perpendicular to its length; and also their points of application. If $z' = z''$, the two forces $\omega^2 . Mx_,$, $\omega^2 . My_,$, are applied to the same point, and are reduced to a single force $\omega^2 . M\sqrt{x_,^2 + y_,^2}$, which expresses the pressure on the axis arising from the motion of the system.

238. *Cor.* If we suppose the axis of rotation to be *one of the three principal axes crossing at the centre of inertia*; then since this centre is in the axis of z, we have $x_, = 0$ and $y_, = 0$, and let $z_, = \alpha$. If we therefore transform the origin to this point, the coordinates of dm are $x, y, z - \alpha$, and hence, by the property of the principal axis,

$$\int x(z - \alpha)\,dm = \int xz\,dm - \alpha\int x\,dm = 0,$$

$$\int y(z - \alpha)\,dm = \int yz\,dm - \alpha\int y\,dm = 0.$$

Now $\int x\,dm = Mx_, = 0$, and $\int y\,dm = My_, = 0$, since the centre of inertia is in the axis of z, and therefore

$$\int xz\,dm = 0, \text{ and } \int yz\,dm = 0.$$

Hence since $\omega^2 . Mx_,$ the resultant of all the forces in the plane x, z, and the sum $\omega^2 \int xz\,dm$ of their moments are nothing, these forces are in equilibrio independently of the fixed axis: similarly the forces in the plane of y, z are in equilibrio with each other; whence, if the system revolve round a principal axis through the centre of inertia, the centrifugal force of the parts will not exert any pressure upon the axis, so that if this axis were suddenly to cease to be fixed, the motion will not undergo any change, but will continue exactly as before.

239. *Cor.* 2. If the axis be *a principal axis*, but do *not* pass through the centre of inertia, then neither $x_{,} = 0$, nor $y_{,} = 0$, but still,

$$\int xzdm = 0, \qquad \int yzdm = 0,$$

whence z' and z'' are each $= 0$, and therefore the axis of rotation undergoes a pressure equal to $\omega^2 . \mathrm{M} \sqrt{x_{,}^2 + y_{,}^2}$, applied at the origin. If, therefore, this point only is fixed, the pressure arising from the centrifugal force will be destroyed; and hence, if we suppose the axis to cease to be fixed and to be free to turn round the origin, the motion will not be changed, but will continue exactly as before.

Hence if there be a fixed point in any system, there are three axes passing through that point, round which the system may revolve uniformly, without displacing them.

240. The principal axes are the only ones to which this property attaches; for if the system be compelled to revolve round any other axis through this point, this axis will undergo a pressure which does not pass through the origin, since z' and z'' are not nothing. If, therefore, this axis is not a fixed axis, but capable of turning round the origin, which alone is fixed, the pressure will not be destroyed, the axis of rotation will consequently be displaced, and the motion will change.

241. If a rigid system with a fixed point be put into motion by any instantaneous or other force, its motion will continue permanent, if it commences round one of the principal axes passing

through that fixed point: hence it will be necessary and sufficient that the effect, which the axis of rotation experiences at the commencement of the motion, should be reducible to a single force perpendicular to this axis, and passing through the fixed point.

242. Prop. *To determine the pressure upon the axis, when the system is put into motion by any instantaneous force.*

Let the axis of z be taken as the axis of rotation, &c. as before. Then if ω is the angular velocity communicated to the system by the force, $r\omega$ is the velocity of any element dm, and $r\omega dm$ its momentum in the direction of the tangent to the circle whose radius is r.

Now in order to apply the principle (205), we must estimate this motion in a direction *contrary* to that in which it acts; and if we then resolve it into three, one parallel to the axis of z, and therefore ineffectual; the other two respectively parallel to the axes of x, and y are

$$-y\omega dm \text{ and } x\omega dm.$$

Similarly resolving the motions communicated to all the other elements, and introducing M, $x_,$ and $y_,$ as before, we have

$$-\mathrm{M}y_,\omega = \int -y\omega dm, \qquad \mathrm{M}x_,\omega = \int x\omega dm:$$

where $-\mathrm{M}y_,\omega$ represents the resultant of all the motions parallel to the axis of x, and $\mathrm{M}x_,\omega$ the resultant of all parallel to the axis of y.

If we then take z' and z'', as before, for the distances of these resultants from the plane of x, y, we have for the sums of the moments of the forces,

$$\mathrm{M}y_,z' = \int yz\,dm, \text{ and } \mathrm{M}x_,z'' = \int xz\,dm,$$

to determine z' and z''.

And also if v represent the velocity which the force would tend to impress in a plane perpendicular to the axis, on a portion of mass equal to μ, we have for the forces in equilibrio by the principle,

$$\mu v, \quad -\mathrm{M}y_{,}\omega, \quad \mathrm{M}x_{,}\omega.$$

243. *Cor.* 1. If $\int yzdm = 0$, and $\int xzdm = 0$, then $z' = z'' = 0$, therefore the forces $-\mathrm{M}y_{,}\omega$, $\mathrm{M}x_{,}\omega$ are in the plane of x, y; and since the plane in which the force acts may be assumed as the plane of x, y, we then have three forces in the same plane in equilibrio round a fixed point which is the origin. The resultant then of these forces may easily be obtained, and we then have the pressure on the axis arising from the force.

244. *Cor.* 2. If the quantities $\int yzdm$, $\int xzdm$ are not nothing, the forces, $-\mathrm{M}y_{,}\omega$, $\mathrm{M}x_{,}\omega$ are not in the plane of x, y. We must therefore reduce them to that plane, and the parts which are in that plane will give the pressure perpendicular to the axis; the other parts will determine a pressure parallel to the axis of rotation.

245. PROP. *To determine how an instantaneous force must act on the system, in order that there may be no pressure from it on the axis:*

The case where $z' = z'' = 0$ is the only one in which this can occur, and for this it is necessary that the resultant of the three forces should be in equilibrio, independently of the fixed point. Hence the three components must separately $= 0$ (18). If then α and β are the angles which the direction of the force makes with the axes,

$$\mu v \cos.\alpha - \mathrm{M}y_{,}\omega = 0, \qquad \mu v \sin.\alpha' + \mathrm{M}x_{,}\omega = 0.$$

Substituting for ω its value $\dfrac{\mu v h}{\int r^2 dm}$ (236) and reducing, we obtain

$$\cos.\ \alpha . \int r^2 dm - M h y_{,} = 0, \quad \cos.\ \beta . \int r^2 dm + M h x_{,} = 0.$$

By these equations, since $\cos.^2 \alpha + \cos.^2 \beta = 1$, we find,

$$(\int r^2 dm)^2 = M^2 h^2 (x_{,}^2 + y_{,}^2),$$

$$\tan.\ \alpha . \frac{y_{,}}{x_{,}} + 1 = 0.$$

If the centre of inertia of the system be supposed projected on the plane of x, y, the tangent of the angle which a line drawn from it to the origin makes with the axis of $x = \frac{y_{,}}{x_{,}}$, hence by the known form of trigonometry this line is perpendicular to the direction of the force. We conclude therefore that *the direction of the force ought to be perpendicular to the plane passing through the fixed axis and the centre of inertia.* The other equation gives us the distance at which it meets the plane, and putting $r_{,} = \sqrt{x_{,}^2 + y_{,}^2}$, we have

$$h = \frac{\int r^2 dm}{M r_{,}}.$$

246. This proposition enables us to determine the point which is called the *Centre of Percussion,* or the point at which a system must be acted on by an *instantaneous* force, in a direction perpendicular to the plane through the axis and the centre of inertia, so as to produce no pressure on the axis of rotation.

DYNAMICS.

SECTION II.

INTRODUCTION.

ON THE LAWS OF THE MOTION OF MATTER.

247. We now proceed to apply the conclusions of the preceding section to the various phenomena of motion exhibited in nature: and first we will consider how far the assumptions which were made (147) with respect to the *inertia* of the elementary point or particle hold good with respect to the different forms of matter.

That the cause of motion does not exist in the body moved, will be readily admitted from daily experience: that the motion, if it does continue, will have a tendency to be rectilinear will be evident, because there can be no reason why the body should deviate to the right rather than to the left of the line; but that it will have a tendency to be uniform is not so evident. It might at first sight be imagined that every body had a natural tendency to return to a state of rest; but experiment shews that in proportion as we remove the impediments to motion, which no experiment can completely do, so we gradually approximate to uniformity of motion as a limit. If we extend our observations beyond the surface of the earth, we see bodies which have for ages maintained a motion which was once communicated to them, without any sensible alteration. Hence we conclude that matter is endued with *inertia*, and we assert as a law of motion, *That every body at rest will continue at rest, and a body in motion will continue in uniform rectilinear motion, unless it is acted on by some external force.*

248. With respect to the forces which produce any change of rest or motion, we may observe that we measure them by their effects (4), and that they are considered as *instantaneously* or *continually* acting, according as the duration of their action is for an indefinitely small time, or continued incessantly during a longer period. And it follows from the property of inertia that the motion communicated by an *instantaneous* or *impulsive* force would, if undisturbed, be *uniform;* but that if the force acted *continually,* the motion would be accelerated or retarded, according as the force conspired with or was opposed to any previous motion. Again, the acceleration or retardation may be regular or variable, in which case the force is an *uniform* or *variable* force (147[a]).

---

[a] The second law of motion is considered as implied in our assumptions (4) as to the method of estimating force. It is usually enunciated thus:

Every motion or change of motion produced in a given body, during a given time, is proportional to the force which produces it, and in the direction of the force.

CHAPTER I.

ON THE MOTION OF HEAVY BODIES NEAR THE SURFACE OF THE EARTH IN VACUO.

249. Experiment shews that if a heavy body fall from rest *in vacuo* near the earth's surface, it will acquire a velocity proportional to the time of its fall: and also that the velocity is independent of the form or mass of the body. Gravity therefore, or the force which causes this fall (59) is an *uniform* force (147), and since it acts equally on every particle of the body, every particle is endued with a parallel motion, and the formulæ for the motion of a point will determine the motion of the whole assemblage, if we can measure the force, or find the velocity generated in the unit of time.

This force, we must observe, is different at different parts of the earth's surface, but for the same or nearly the same place may be considered as constant (60); and the latest experiments teach us that in the latitude of London a body will fall in vacuo through $16\frac{1}{10}$ feet nearly in 1″ of time. Hence, since the body would move through double this space in the same time with the velocity acquired continued uniform (182), the velocity generated in 1″ would be measured by $32\frac{1}{5}$ feet, which quantity we will represent by g.

250. *Prop. To determine the motion of a heavy body near the earth's surface.*

If we substitute g for f in the formulæ for the motion of a point by the action of an uniform force, deduced in art. 176, &c. we obtain formulæ for the solution of all questions relating to the fall of bodies, viz.

$$v = v' \pm gt,$$

$$s = v't \pm \tfrac{1}{2}gt^2,$$

$$2gs = v^2 \backsim v'^2;$$

where v' denotes the velocity which the body has at the commencement of the motion, whether by an impulsive force or any previous action of the same force.

If the body moves from rest, these formulæ become

$$v = gt, \quad s = \tfrac{1}{2}gt^2, \quad v^2 = 2gs.$$

251. If the body be projected vertically upwards, gravity becomes an uniformly retarding force, and the formulæ are still applicable, if we substitute velocity destroyed for velocity generated, &c. Thus, the body will rise until the whole velocity of projection is destroyed, and the height to which it will rise is equal to that through which it must fall to acquire a velocity equal to the velocity of projection; and would therefore be determined from the formula $v^2 = 2gs$.

Similarly its position at any time if the velocity of projection were given would be determined from the equation,

$$s = v't - \tfrac{1}{2}gt\ .$$

And so also for other cases.

252. *Prop. To determine the motion of a heavy body projected obliquely to the horizon.*

We will here suppose that no rotation takes place, or rather we will determine the motion of the body as if it were collected in its centre of gravity.

It is evident that the path of the body will lie in the vertical plane which passes through the direction of projection, and if we therefore, with this point as origin, take an horizontal line as the axis of x, and a vertical line as axis of y, two of the general equations (151) are sufficient, viz.

$$X = \frac{d^2x}{dt^2}, \qquad Y = \frac{d^2y}{dt^2}.$$

But since the only accelerating force acting on the body is gravity which tends to draw the body down, we have

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g;$$

whence by integration we obtain,

$$x = at, \qquad y = -\tfrac{1}{2}gt^2 + bt,$$

in which operations we must observe, that we have assumed the origin of the time to coincide with the origin of coordinates.

If we then eliminate t, we have,

$$y = \frac{b}{a} \cdot x - \tfrac{1}{2}g \cdot \frac{x^2}{a^2}.$$

In order to determine the constants introduced at the first integration, let v be the velocity of projection and ϵ the angle which the direction of projection makes with the horizontal axis, or the angle of elevation; then

v cos. ϵ is the velocity in the axis of x,

v cos. ϵ the velocity in the axis of y.

$$\text{but } \frac{dx}{dt} = a, \text{ and } \frac{dy}{dt} = -gt + b,$$

and, therefore, if we make $t = 0$, we have (151)

$$a = v \text{ cos. } \epsilon, \qquad b = v \text{ sin. } \epsilon.$$

Substituting therefore these values for a and b, we obtain

$$y = x \tan.\epsilon - \frac{g}{2v^2 \cos.^2\epsilon} \cdot x^2,$$

which is the equation to the path of the projectile.

If h represent the height *due* to the velocity of projection we have (181) $v^2 = 2gh$,

$$\therefore\ y = x \tan.\epsilon - \frac{x^2}{4h \cos.^2\epsilon}.$$

This is evidently the equation to a *parabola* referred to horizontal and vertical axes through any point, and whose principal parameter, or latus rectum, is $4h \cos.^2\epsilon$, ϵ being the angle which the tangent at the origin makes with the axis of x.

253. *Cor.* 1. The *horizontal range*, or the distance at which the projectile again meets the horizontal plane, is easily found by making $y = 0$; and we thus find

$$x = 4h \sin.\epsilon \cos.\epsilon = 2h \sin. 2\epsilon = \textit{Range.}$$

254. *Cor.* 2. The *greatest height* above the horizontal plane is determined by finding the maximum ordinate: and by differentiating we have

$$\frac{dy}{dx} = \tan.\epsilon - \frac{x}{2h \cos.^2\epsilon} = 0$$

$$\therefore\ x = 2h \sin.\epsilon . \cos.\epsilon = h \sin. 2\epsilon.$$

Hence the abscissa corresponding to the greatest height is half the range: and substituting the value of x in the equation, we find

$$y = h \sin.^2\epsilon = \textit{Greatest height.}$$

255. *Cor.* 3. The *time of flight* is easily determined by
tuting in the equation $x = at$, or rather $x = v$

If the body descend down the whole length, then $s = l$ and the last two expressions give us,

$$t = l. \sqrt{\frac{2}{gh}}, \qquad v^2 = 2gh;$$

which last agrees with the general result in the last article.

260. If in a circle, whose diameter is vertical, chords be drawn to either extremity, as PA or PB (fig. 47.), the time down either of these chords is equal to the time down the diameter, wherever the point P be taken in the circumference.

For $PB^2 = AB.\ BM = 2rh$

$$\therefore\ t^2 = \frac{2l^2}{gh} = \frac{2}{g}(2r).$$

The velocity also varies as the length, since $h \propto l^2$.

261. *To determine the motion when one body attached to a string passing over a wheel on the top of an inclined plane draws another up the plane.*

The string on the side of the body drawn up (p') is supposed to be parallel to the plane (fig. 48.), and let p be the weight, hanging freely.

Then the *moving force* of the system (200) is here the difference of the moving force of its parts; and if h and l represent the height and length as before,

$$\text{the whole moving force} = \left(p - p'\frac{h}{l}\right) g.$$

The accelerating force, therefore, which equals the moving force divided by the mass (200), is

$$\frac{pl - p'h}{l(p + p)} . g.$$

Substituting this value for f in the formulæ for motion by the action of the uniform force, deduced in 176, &c. we can determine the time, velocity acquired, &c.

262. Similarly, if two planes are placed back to back, and a body on one draws up another body on the other by means of a string over a wheel on the common vertex, as in fig. 37, it may be shewn that the accelerating force of the system is

$$\frac{(pl' - p'l)h}{ll'(p + p')} \cdot g.$$

263. **Prop.** *To determine the motion of a body descending by its own weight down any arc of an inverted cycloid, with its base horizontal and axis vertical.*

The *velocity* we have seen depends solely on the distance fallen through in the vertical line (258).

The *time* can be determined from the general equation (190),

$$dt = \frac{ds}{\sqrt{(2f(h - x))}},$$

in which x is measured on the axis in the direction of which the force acts, and h is the abscissa of the point K (fig. 49.) at which the motion commences.

If, then, the cycloid be referred to vertical and horizontal axes through the lowest point, and if its axis $= 2a$, we obtain from the equation to the curve, putting $AM = x$, $PM = y$, $AP = s$,

$$ds = dx. \sqrt{\frac{2a}{x}}.$$

Substituting this value with its proper sign, and also g for f in the equation above, we find

$$dt = \sqrt{\frac{a}{g}} \cdot \frac{-dx}{\sqrt{(hx - x^2)}}.$$

Whence, integrating, and observing that when $t = 0$, $x = h$,

$$t = \sqrt{\frac{a}{g}}\left(\pi - \text{versin.}^{-1}\frac{2x}{h}\right).$$

And therefore, for the time of descent to the lowest point, or when $x = 0$

$$t = \pi\sqrt{\frac{a}{g}}.$$

264. *Cor.* 1. We here observe, that the time of descent to the lowest point, is independent of the point at which the motion commences, and is therefore the same, whatever point in the arc be taken. On account of this remarkable property, the cycloid is called *tautochronal.*

265. *Cor.* 2. Since the velocity of the vertex is that due to falling through the altitude $h = \text{A H}$, it will carry the body through the opposite arc until the whole velocity is destroyed, or until the body has ascended to the point k, which has the same vertical abscissa. When it has reached this point, it will stop, and again commence descending; it will retrace the same path, and return to the point K, from which it first began to descend; and so on continually.

The time from K to k will $= 2\pi\sqrt{\frac{a}{g}}$.

266. PROP. *To determine the* time *of descent through a circular* arc.

In this case, $ds = \frac{a\,dx}{\sqrt{(2ax - x^2)}}$, and we have therefore to integrate

$$dt = \frac{a}{\sqrt{2g}} \cdot \frac{-dx}{\sqrt{((h-x)(2ax-x^2))}}.$$

But this equation can only be integrated by means of a series; and we can obtain a converging series by writing the equation, thus,

$$dt = -\frac{1}{2}\sqrt{\frac{a}{g}} \cdot \frac{dx\left(1-\frac{x}{2a}\right)^{-\frac{1}{2}}}{\sqrt{(hx-x^2)}}.$$

Expanding the expression in the numerator by the binomial theorem, &c. we have

$$dt = -\frac{1}{2}\sqrt{\frac{a}{g}} \cdot \frac{dx}{\sqrt{(hx-x^2)}} \left\{1 + \frac{1}{2} \cdot \left(\frac{x}{2a}\right) + \frac{1.3}{2.4} \cdot \left(\frac{x}{2a}\right)^2 + \frac{1.3.5}{2.4.6} \cdot \left(\frac{x}{2a}\right)^3 + \&c. \right\}$$

Thus the terms to be integrated are of the form $\frac{-x^n dx}{\sqrt{(hx-x^2)}}$ the exponents n being natural numbers beginning with 0. If, then, the operation[a] be performed, and the integral taken between the limits $x = h$, and $x = 0$, we find

$$t = \frac{\pi}{2} \cdot \sqrt{\frac{a}{g}} \cdot \left\{1 + \left(\frac{1}{2}\right)^2 \cdot \left(\frac{h}{2a}\right) + \left(\frac{1.3}{2.4}\right)^2 \left(\frac{h}{2a}\right)^2 + \left(\frac{1.3.5.}{2.4.6.}\right)^2 \left(\frac{h}{2a}\right)^3 + \&c. \right\}$$

---

[a] See Lardner's Integral Calculus, 222.

267. *Cor.* 1. If h be very small when compared with a, the first term of this series will give us an approximate value for the time, viz.

$$t = \frac{\pi}{2}\sqrt{\frac{a}{g}},$$

which is the same as the expression for the time down the arc of a cycloid whose axis is $\frac{1}{4}$ the radius of the circle. Hence it appears also, that the time of descent down a very small circular arc is constant, like the time down any cycloidal arc.

268. From this beautiful property of *tautochronism* which attaches to the cycloid, it was supposed that a pendulum made to oscillate in this curve, would have peculiar advantages. The method of accomplishing this is naturally suggested by the property of the *evolutes:* and as a cycloid has for its evolute another equal cycloid, we have only to take two semicycloids, and to place the extremities of their bases contiguous, in the same horizontal line, as in fig. 50. If we then suspend a body at the point O, common to the two semicycloids, by a flexible string equal in length to either semicycloid, or, which is the same thing, to twice the axis, and make it oscillate between the two, it will generate the involute, or another equal cycloid.

The constant change however in the centre of motion arising from the contact of the string with the two curves or cheeks, which changes also the relative velocity of the different parts of the vibrating body, renders this contrivance although beautiful in theory, yet useless in practice; independently of the difficulty of obtaining a string sufficiently flexible, and of ensuring accuracy in the plates.

Vibrations therefore in small circular arcs, which are at the same time also most natural, have been adhered to in practice, and we have shewn (267) that as long as the arc is small, these vibrations have all the advantages of the vibrations in the arcs of a cycloid.

269. In the preceding investigations we have supposed that the velocity &c. of every particle of the body was the same, or the body to be concentrated in one point. And we have also supposed that the string itself by which the body was suspended was perfectly flexible and without weight, or in other words, we have been discussing the oscillations of *a Simple Pendulum.*

In order to apply these deductions to a pendulum consisting of a heavy rod or chain, and a body of finite parts, or to a *Compound Pendulum,* we must determine a point which is called the *Centre of Oscillation,* or the point in any oscillating body, in which, if the whole mass could be concentrated, and suspended by a string without weight, the vibrations of this imaginary simple pendulum would be performed in the same time as the vibrations of the compound one.

270. Prop. *To determine the centre of oscillation of any body.*

Since every particle of the body vibrates in the same vertical plane, the motion is evidently the same as the motion round a fixed axis; and the investigation of such motion gave us the equation (235),

$$\frac{d\omega}{dt} = \frac{\int r\mathrm{F}\cos.\delta.dm}{\int r^2 dm},$$

to determine the angular velocity, when the force F acting on any element dm, and the angle δ, made by the direction of the force with the tangent to the circle which the element is compelled to describe, are given for every element.

In this case where gravity is the only force which acts on the system, and the axis of rotation is a horizontal line, let us assume the axis of z which is the axis of rotation, and also the axis of y as horizontal, and the axis of x vertical, and therefore in the direction of gravity.

Then, if m be the position of any particle (fig. 51), $m\,\mathrm{T}$ be a tangent to the circle described with $\mathrm{A}\,m$ as radius, $m\,\mathrm{D}$ perpendicular to the axis of x, and $m\,\mathrm{H}$ parallel to it,

$$\therefore\ y = m\,\mathrm{D} = \mathrm{A}\,m.\ \cos.\ \mathrm{A}\,m\,\mathrm{D}$$

$$= \mathrm{A}\,m.\ \cos.\ \mathrm{T}\,m\,\mathrm{H} = r \cos.\ \delta.$$

Hence the general formula becomes, since also $\mathrm{F} = g$.

$$\frac{d\omega}{dt} = \frac{\int g y\, d m}{\int r^2 dm} = \frac{g \mathrm{M} y_{,}}{\int r^2 dm},$$

if M = whole mass of the body, and $y_{,}$ = the distance of its centre of gravity from the plane of x, z.

But if a = the distance of this point from the axis of rotation, $\mathrm{M}\,a^2$ is the moment of inertia of the system collected in its centre of gravity about this axis, and if $\mathrm{M}\,k^2$ be taken for the moment about an axis drawn through the centre of gravity parallel to $\mathrm{A}z$, then $\mathrm{M}\,(k^2 + a^2)$ is the moment of the body round the axis of rotation (231). Hence

$$\frac{d\omega}{dt} = \frac{g y_{,}}{k^2 + a^2}.$$

And since the centre of gravity of the system describes a circular arc as well as every other point, let (fig. 52.) G be the centre of gravity, G T a tangent, A G = a, &c.

$$\therefore\ y_{,} = a \cos.\ \delta \text{ and } \frac{d\omega}{dt} = \frac{g\,a \cos.\ \delta}{k^2 + a^2}.$$

Let now a simple pendulum whose length = l, be made to oscillate, then since here we have only one particle m, the general expression becomes in this case

$$\frac{d\omega}{dt} = \frac{g\,l\,m.\cos.\theta}{l^2 m} = \frac{g\cos.\theta}{l}.$$

Hence, in order that this may oscillate in the same time and in the same manner as the compound pendulum, it ought to have at any instant the same position with regard to the axis of vibration, and the same angular velocity at that instant, hence the two values of $\frac{d\omega}{dt}$ ought to be always equal and θ also the same in both cases,

$$\therefore\ l = \frac{k^2 + a^2}{a}.$$

If therefore, in the plane passing through the axis of suspension and the centre of gravity of the body, a straight line be drawn parallel to the axis at a distance l, all the points of the body which are on that straight line are so many *centres of oscillation.*

The condition that the line should be drawn in the plane through the axis and the centre of gravity is necessary in order that the angle represented by θ may be the same in the two expressions compared above.

271. *Prop. The axis of the centres of oscillation and the axis of suspension are reciprocal*: that is, if the axis of the centres of oscillation be made the axis of suspension, the axis of suspension will become the axis of the centres of oscillation.

For let l', a', represent with regard to the new axis the same quantities as l, a, represent with regard to the former axis of suspension. Then

$$l' = \frac{k^2 + a'^2}{a'}.$$

But since l is the distance of the two axes,

$$l = a' + a, \quad \therefore a' = l - a.$$

Also from the value of l in terms of k and a, we find

$$l - a = \frac{k^2}{a} \qquad \therefore a' = \frac{k^2}{a}.$$

Hence, substituting for a' we obtain

$$l = \frac{k^2 + \frac{k^4}{a^2}}{\frac{k^2}{a}} = \frac{k^2 + a^2}{a} = l.$$

272. In the preceding articles we have supposed the force of gravity to be *uniform,* and we have observed that this is not true unless the motion be near the surface of the earth, and our observations are confined to the same place. If a body be supposed to fall towards the earth from a considerable distance above it, we must introduce the law of the variation of gravity, in order to determine its motion. Experiment teaches us that the force varies inversely as the square of the distance from the earth's centre, and hence we can apply the formulæ deduced in 171, &c.

273. Prop. *To determine the motion of a body falling from a given distance towards the earth, the force of gravity not being constant.*

Let ϕ = the force of gravity at the unit of distance,

a = the distance of the point at which the motion commences,

x = any variable distance.

$$\therefore f = \frac{\phi}{x^2}.$$

Hence by substituting this value of f in the formula (173), and integrating, we find

$$v^2 = 2\phi\left(\frac{1}{x} - \frac{1}{a}\right),$$

which gives us the velocity at any distance x.

If $x = 0$ $\therefore$ $v^2 = \infty$, or velocity at the centre would be infinite if the same law of force continued.

Hence we also find by substitution in the formula (174) and integrating

$$t = \left(\frac{a}{2\phi}\right)^{\frac{1}{2}} \left\{ (a\,x - x^2)^{\frac{1}{2}} + \frac{a}{2}\left(\pi - \text{versin.}^{-1}\frac{2x}{a}\right) \right\}.$$

$$\text{If } x = 0 \quad \therefore \; t = \frac{\pi\, a^{\frac{3}{2}}}{2(2\phi)^{\frac{1}{2}}};$$

or the square of the time of falling to the centre varies as the cube of the distance fallen through.

CHAPTER II.

ON THE REVOLUTION OF THE HEAVENLY BODIES ROUND THE SUN.

274. The phenomena of the heavenly bodies may be divided into three classes: the first comprehending the motion of revolution round the sun; the second, the motion of rotation round their centres of inertia; and the third, their figure and the oscillations of the fluids on their surfaces. It is only the most simple of the first of these classes which we propose at present to consider: viz. the force which produces the motion of revolution, and the orbits which the bodies would, if undisturbed, describe.

Observation has established three laws respecting the motion of the planets, which from their discoverer are called Kepler's laws;

1st. *The planets move in plane curves, and their radii vectores describe round the sun areas proportional to the time of their description.*

2nd. *The orbits of the planets are ellipses with the centre of the sun in one of the foci.*

3rd. *The squares of the times of revolution of the different planets are to one another as the cubes of the major axes of the orbits.*

These laws we must observe relate only to the motion of the centre of gravity of each planet; and we must therefore limit our conclusions to the motion of that point, or, in other words, we shall only determine the motion of *translation* of the planets.

275. From the *first* of these laws we deduce by means of art. 158, that the accelerating force thus applied to the centre of gravity is always *directed in the line which joins this centre and the centre of the sun.*

276. From the *second* law we are able to discover the law of the *variation* of this force.

For since the general equation to an ellipse referred to the focus as a pole is

$$r = \frac{a(1 - e^2)}{1 + e \cos.(\omega - \alpha)}$$

in which a is the semimajor axis or as it is sometimes called the *mean distance,* and e the eccentricity. The angle α is the angle contained between a fixed line and the least distance, or the *longitude of the perihelion;* the least distance being called the *perihelion,* and the greatest the *aphelion* distance. The angle $\omega - \alpha$, which expresses the angular distance of the planet from the perihelion, is called the *true anomaly.*

By differentiating this expression we find

$$\frac{dr}{r^2 d\omega} = \frac{e \sin.(\omega - \alpha)}{a(1 - e^2)};$$

then squaring both sides, and eliminating the angle by the equation, we obtain

$$\frac{dr^2}{r^4 d\omega^2} = \frac{2}{a(1 - e^2)} \cdot \frac{1}{r} - \frac{1}{r^2} + \frac{1}{a^2(1 - e^2)}.$$

Now the general equation for determining the force is (168)

$$c^2 \left(\frac{dr^2}{r^4 d\omega^2} + \frac{1}{r^2}\right) + 2 \int F dr = C.$$

Substituting therefore the value obtained for $\frac{dr^2}{r^4 d\omega^2}$, and differentiating, we have,

$$F = \frac{c^2}{a(1 - e^2)} \cdot \frac{1}{r^2}.$$

Hence we conclude that the force, which acts upon the planets, varies inversely as the square of the distance from the sun. By such a force, therefore, combined with an original impulse, the planets are retained in their orbits, and perform their revolutions in space.

The constant factor $\frac{c^2}{a(1 - e^2)}$ expresses the intensity of this force at the unit of distance, and is evidently constant for the same planet.

277. From the *third* law we are able to prove that the *intensity* of the force is the same for all the planets.

The quantity c is the double of the area described in the unit of time (164), and if therefore T is the time of one revolution of a planet, cT will be the double of the area described by the radius vector.

But the area of an ellipse $= \pi a^2 \sqrt{(1 - e^2)}$, and therefore

$$c\mathrm{T} = 2\pi a^2 \sqrt{(1 - e^2)},$$

$$\therefore \frac{c^2}{a(1 - e^2)} = \frac{4\pi^2 a^3}{\mathrm{T}^2}.$$

Similarly, if a', e', c', T$'$, expressed the same quantities in any other orbit, we should have,

$$\frac{c'^2}{a'(1 - e'^2)} = \frac{4\pi^2 a'^3}{\mathrm{T}'^2}.$$

But by Kepler's third law,

$$T^2 : T'^2 :: a^3 : a'^3;$$

and therefore

$$\frac{c^2}{a(1-e^2)} = \frac{c'^2}{a'(1-e'^2)}.$$

Hence we conclude that this force does not vary from one planet to another, except in consequence of change of distance; and hence we deduce another analogy between this force and the force of gravity.

278. We will next proceed to the *inverse* problem, and investigate the orbit which a body will describe when the force varies inversely as the square of the distance.

The general formula for this investigation is (168),

$$c^2\left(\frac{du^2}{d\omega^2} + u^2\right) - 2\int \mathrm{F}\frac{du}{u^2} = \mathrm{C},$$

in which $u = \frac{1}{r}$.

If then the force at the unit of distance $= \phi$,

$$\therefore \mathrm{F} = \phi u^2 \quad \therefore \int \mathrm{F}\frac{du}{u^2} = \phi u.$$

Hence the general formula becomes

$$c^2\left(\frac{du^2}{d\omega^2} + u^2\right) - 2\phi u = \mathrm{C},$$

which is the differential equation to the orbit.

To integrate this, we find

$$d\omega^2 = \frac{c^2\,du^2}{\mathrm{C} + 2\phi u - c^2u^2},$$

and therefore substituting $u = z + \frac{\phi}{c^2}$, and writing b^2 for $\frac{1}{c^4}(Cc^2 + \phi^2)$, we have

$$d\omega = \frac{\pm dz}{\sqrt{(b^2 - z^2)}};$$

whence, by integration

$$\omega = k \mp \cos.^{-1} \frac{z}{b},$$

$$\therefore z = + b \cos. (\omega - k).$$

The positive sign being taken on account of the arbitrary constant k, which is equivalent to increasing the angle by two right angles. Then introducing u and r, we find

$$u = \frac{\phi}{c^2} + b \cos. (\omega - k),$$

$$\therefore r = \frac{c^2}{\phi + bc^2 \cos. (\omega - k)},$$

which is the equation to a conic section referred to the focus.

In order to determine the constants, and therefore the exact orbit described, some other circumstances must be given, and we will now determine the elements of the orbit, when the initial velocity, distance, and angle of projection are given.

Let α = angle made by the direction of projection with the distance,

ρ = the given distance,

v' = the velocity of projection:

Then, since the sine of the angle which the tangent makes with the radius vector $= \frac{r d\omega}{ds}$, ds being the element of the curve, we have $\sin.\alpha = \frac{\rho\, d\omega}{ds}$, hence

$$c = \frac{\rho^2 d\omega}{dt} = \rho . \frac{ds}{dt} . \frac{\rho\, d\omega}{ds} = \rho v \sin.\alpha.$$

Again, by the equation in art. 166,

$$v^2 = \mathrm{C} - 2\int \mathrm{F}\, dr = \mathrm{C} + \frac{2\phi}{r},$$

therefore we have,

$$\mathrm{C} = v'^2 - \frac{2\phi}{\rho}.$$

Also, since by comparing the result of the integration with the general equation to the conic section, we find

$$a\,(1 - e^2) = \frac{c^2}{\phi}, \text{ and } e = \frac{bc^2}{\phi} = \frac{1}{\phi} \surd(\mathrm{C}c^2 + \phi^2),$$

$$\therefore a = -\frac{\phi}{\mathrm{C}}, \quad e = \surd\left(1 + \mathrm{C}.\frac{c^2}{\phi^2}\right).$$

by substituting the values just found for C and c, we have

$$a = \frac{\phi}{\frac{2\phi}{\rho} - v'^2}, \text{ and } e^2 = 1 + \frac{\rho^2 v'^2 \sin.^2\alpha \left(v'^2 - \frac{2\phi}{\rho}\right)}{\phi^2}.$$

The value of a, thus found, is independent of the angle of projection; whence it appears, that in whatever direction the body is projected, the axis major of the orbit will be the same, if the velocity remain unaltered.

The orbit will be an ellipse, hyperbola, or parabola, according as e is less, greater than, or equal to unity, or according as the quantity

$$\varrho^2 v'^2 \sin.^2\alpha \left(v'^2 - \frac{2\phi}{\varrho}\right)$$

is negative, positive, or equal to nothing; that is, according as v'^2 is less, greater than, or equal to $\frac{2\phi}{\varrho}$.

279. We will next investigate the *time* of describing any portion of the orbit.

By the general expression in art. 167, we have,

$$\frac{dr^2}{dt^2} + \frac{c^2}{r^2} + 2\int F dr = C.$$

Therefore, in this case

$$\frac{dr^2}{dt^2} + \frac{c^2}{r^2} - \frac{2\phi}{r} = C.$$

The values of the constants are easily found in terms of the elements of the orbit from the investigations in the last article, and we have

$$C = -\frac{\phi}{a}, \qquad c^2 = \phi a(1 - e^2).$$

Substituting, therefore, these values, and multiplying by r^2, we obtain,

$$\frac{a}{\varphi} \cdot \frac{r^2 dr^2}{dt^2} + (a - r)^2 - a^2 e^2 = 0,$$

and hence

$$dt = \sqrt{\frac{a}{\phi}} \cdot \frac{r\,dr}{\sqrt{(a^2 e^2 - (a - r)^2)}}.$$

In order to integrate this expression, let $a - r = aez$,

$$\therefore\ dt = -a\sqrt{\frac{a}{\phi}}\cdot\frac{(1-ez)dz}{\sqrt{(1-z^2)}},$$

$$\therefore\ t = a\sqrt{\frac{a}{\phi}}\left\{\cos.^{-1}z - e\sqrt{(1-z^2)}\right\} + \mathrm{K}.$$

When $z = 1$, or $r = a - ae$, and the body is at the nearer apsis, let t be t'

$$\therefore\ t' = \mathrm{K}.$$

Hence $t - t' = a\sqrt{\frac{a}{\varphi}}\left\{\cos.^{-1}z - e\sqrt{(1-z^2)}\right\}$

in which $t - t'$ is the time through any portion of the arc from the nearer apsis.

If $r = a(1+e)$ $\therefore$ $z = -1$, and we have the time from the nearer to the farther apsis, or

the time of half a revolution $= a\sqrt{\frac{a}{\varphi}}\cdot\pi$

Hence if T be the periodic time

$$\mathrm{T}^2 = \frac{4\pi^2a^3}{\phi}.$$

CHAPTER III.

ON THE IMPACT OF BODIES.

280. If two bodies come into contact with one another, and if their centres of gravity move in the same straight line, which is perpendicular to a plane touching both bodies in the point of contact, the impact is said to be *direct* and *central.*

If the straight line described by the centre of gravity of one of the bodies, is perpendicular to the tangent plane, but does not pass through the centre of gravity of the other, the impact is said to be *direct,* but *eccentric.*

If the path of the centre of gravity of one of the bodies is not perpendicular to the tangent plane, the impact is said to be *oblique.*

281. In investigating the principles of the impact of bodies, we *suppose* that there are certain bodies which, by impact, suffer compression, and, after impact, endeavour to recover their original form with a force bearing some certain ratio to the impact: so that this force of restitution gives them, in a contrary direction, a determinate part of the force, and therefore of the velocity, which they had lost by the impact.

This force of restitution is called the *elasticity,* which is said to be *perfect,* if the force of restitution is always equal to the force lost in the impact, and *imperfect,* if it is less.

If the bodies suffer no compression they are called *hard,* and if, when compressed, they exert no force to recover their form, they are called *soft.*

There are no bodies known which are either perfectly elastic or perfectly hard; but these states may be considered as limits to the various degrees of elasticity presented in nature.

282. Prop. *To determine the motion after the* direct *impact of two hard bodies.*

Let m and m' be the two bodies moving in the same direction with velocities v and v', then, if m overtake and impinge on m', the two bodies will, in consequence of the impact, acquire an equal velocity: for m will continue to impart to m' a portion of its velocity, until m' having acquired a velocity equal to m, would no longer impede the motion of m.

Let then x be this common velocity, and since the moving forces before or the *impressed* forces are

$$mv \qquad \text{and} \quad m'v',$$

and the *actual* forces, mx and $m'x$,

we have by d'Alembert's principle (204),

$$mv - mx + m'v' - m'x = 0$$

$$\therefore x = \frac{mv + m'v'}{m + m'}.$$

283. Prop. *To determine the motion after the* direct *impact of two elastic bodies.*

If e represent the portion of the velocity which is recovered by the force of restitution in a contrary direction, and if m, m', v, v' be taken as before, let x represent the common velocity which they would have if hard.

Then by the impact m would lose the velocity $v - x$, and it would also lose by the elasticity $e(v - x)$, hence the whole loss of m will be

$$v - x + e(v - x) = (1 + e)(v - x).$$

Substituting, therefore, for x its value found in the last article, we have

$$\text{the loss of } m = (1 + e)(v - v')\frac{m'}{m + m'},$$

$$\text{and similarly the gain of } m' = (1 + e)(v - v')\frac{m}{m + m'}.$$

284. *Cor.* 1. If $e = 1$ we have the case of *perfect* elasticity; if $e = 0$, we have the case of the last article, &c.

285. *Cor.* 2. The difference of the actual velocities after impact is $e(v - v')$. Hence if the bodies are perfectly elastic, the relative velocity before and after impact is the same, &c.

286. *To determine the motion of two bodies after* oblique *impact.*

Resolve the velocity of each body into two, the one perpendicular to the plane touching both the masses in the point of contact, the other parallel to it. The first of these two occasions the impact, and the effects may be determined as in *direct* impact; the other remains unaltered. Compounding therefore this latter with the results of the former, we shall have the velocities and directions of the two bodies after impact.

287. Similarly, we can determine the effects of the impact of any body on an impenetrable and immoveable plane.

Let v be the velocity of impact,

ι the angle contained by the direction of impact, and a perpendicular to the plane at the point of impact, or angle of incidence,

e the part of the velocity acquired by the elasticity, in a contrary direction.

Then resolving v into two, one perpendicular to the plane, $v \cos. \iota$, and the other parallel, $v \sin. \iota$; the perpendicular one is the only one which causes the impact, and the body will rebound perpendicularly with a velocity $= e\,v \cos. \iota$.

Compounding then this velocity with the one parallel to the plane we have the velocity of reflection (v'),

$$v \sqrt{(\sin^2 \iota. + e^2 \cos.^2. \iota)}.$$

If ρ be the angle of reflection, we also have

$$\tan. \rho = \frac{v \sin. \iota}{ev \cos. \iota} = \frac{1}{e}. \tan. \iota.$$

Hence also we obtain, $v \sin. \iota = v' \sin. \rho$.

288. *Cor.* 1. If the body is perfectly elastic or $e = 1$, the velocity and angle of reflection are equal to the velocity and angle of incidence.

289. *Cor.* 2. If the body is hard, or $e = 0$, the angle of reflection is a right angle, and the body moves along the plane, and $v' = v \sin. \iota$.

290. Prop. *To determine the motion of two non-elastic bodies after* eccentric *impact.*

Let A P B and C P D be sections of the two bodies made by a plane passing through their respective centres of gravity G and g, and through the direction G E of the motion of the first; let P be the point of impact; let the distance of the direction G E from the centre of gravity of the other body be g K $= h$, and let x be the velocity remaining to the first after impact.

Then the other body will have, after impact, two motions, one progressive and the other rotatory, since the direction of the impact does not pass through the centre of gravity (217 and 218). The rotation will take place about an axis through G perpendicular to the section; and let u be the progressive velocity, and ω the angular velocity of the rotation.

First by d'Alembert's principle (204), the forces $m v$, $m' v'$, are in equilibrio with $- m x$, and $- m' u$,

$$\therefore\ m(v - x) + m'(v' - u) = 0.$$

Again, the second body is urged to revolve, by the force which the first body loses in the impact, viz. $m(v - x)$. Hence by the formula deduced in art. 236.

$$\omega = \frac{m h (v - x)}{c},$$

if c represents the momentum of inertia of the mass m'.

Lastly, it appears (282) that as much velocity ought to remain to the body A P B as those points of the other body acquire which lie in the line G E; hence the remaining velocity of the impinging body must be that which the point K, of the other body, acquires. Now the remaining velocity of the impinging body is x, and the velocity which the point K acquires is the velocity arising from the direct impact + the velocity which it acquires from the rotation round G, and therefore it is $u + h\omega$.

Wherefore

$$x = u + h\omega.$$

We have therefore three equations, to determine the three unknown quantities x, u, and ω.

291. *Cor.* 1. The problem is scarcely more difficult if the bodies are elastic: since the force of elasticity restores to the bodies in a contrary direction, a velocity proportional to that which they lose from the impact. We have then to find the three quantities x, u, and ω, as if there were no elasticity; and then to substitute for them respectively,

$$x - e(v - x); \qquad u - e(v' - u);$$

$$\omega + e\omega.$$

292. If the body struck is connected by an inflexible line, with an immoveable axis, it cannot have or acquire any progressive motion. We have then in this case only two unknown quantities, x and ω; to determine which we have the equations,

$$\omega = \frac{mh(v - x)}{c}, \quad \text{and } x = h\omega,$$

in which h and c must now be referred to the axis of rotation.

dicular to the section; and let u be the progressive velocity, and ω the angular velocity of the rotation.

First by d'Alembert's principle (204), the forces $m v$, $m' v'$, are in equilibrio with $-mx$, and $-m'u$,

$$\therefore\ m(v - x) + m'(v' - u) = 0.$$

Again, the second body is urged to revolve, by the force which the first body loses in the impact, viz. $m(v - x)$. Hence by the formula deduced in art. 236.

$$\omega = \frac{m h (v - x)}{c},$$

if c represents the momentum of inertia of the mass m'.

Lastly, it appears (282) that as much velocity ought to remain to the body A P B as those points of the other body acquire which lie in the line G E; hence the remaining velocity of the impinging body must be that which the point K, of the other body, acquires. Now the remaining velocity of the impinging body is x, and the velocity which the point K acquires is the velocity arising from the direct impact + the velocity which it acquires from the rotation round G, and therefore it is $u + h\omega$.

Wherefore

$$x = u + h\omega.$$

We have therefore three equations, to determine the three unknown quantities x, u, and ω.

291. *Cor.* 1. The problem is scarcely more difficult if the bodies are elastic: since the force of elasticity restores to the bodies in a contrary direction, a velocity proportional to that which they lose from the impact. We have then to find the three quantities x, u, and ω, as if there were no elasticity; and then to substitute for them respectively,

APPENDIX.

See Note, page 4.

The following rigorous demonstrations of the parallelogram of forces are added in the form of an Appendix, in order not only to remove any difficulties from the commencement, but also to enable the student to examine the chief of the different demonstrations that have been brought forward. They are those of Laplace, Pontécoulant, and Poisson, nearly verbatim.

I.

To the first two propositions in the text may be added the following, which are readily deduced from the method of estimating force (4), and from the definition of the word *resultant* (8).

1. The resultant of two forces applied to the same point and acting in the *same* direction, is itself in the direction of the forces and equal to their sum; or $R = p' + p''$.

2. The resultant of two forces applied to the same point, and acting in *opposite* directions, is itself in the direction of the greater force and equal to their difference; or $R = p' \backsim p''$.

3. **Prop.** *To find the magnitude and direction of the resultant of two forces, acting on the same point, in directions* at right angles *to each other.*

Let x, y, r represent the magnitudes of the forces and their resultant,
θ the angle made by the direction of the resultant with that of x:

must destroy each other; the two first are in the same direction and make up the resultant; hence

$$r^2 = x^2 + y^2;$$

or *the resultant is represented* IN MAGNITUDE *by the diagonal of the rectangle, the sides of which represent the forces.*

In order to determine the angle θ, suppose the force x to increase, and y to remain constant; then the angle θ will diminish. And if we also suppose dx, the increment of x, to be resolved into dx' and dx'', the first in the direction of the resultant, and the other at right angles to it, the point is acted on by two forces $r + dx'$ and dx'' at right angles to each other, and the resultant of these forces, which we will call r', will make with dx'' an angle equal to $\frac{\pi}{2} - d\theta$ (see fig. 2, App.); hence, by the preceding,

$$dx'' = r'.\,\varphi\left(\frac{\pi}{2} - d\theta\right).$$

And since dx'' is very small when compared with r', the function $\varphi\left(\frac{\pi}{2} - d\theta\right)$ must be very small, and of the form $- k\,d\theta$, k being a constant independent of the angle, and $d\theta$ being negative when dx'' is positive. Hence

$$dx'' = - kr'd\theta = - krd\theta,$$

since r' is very nearly equal to r.

Again, since dx'' makes with dx an angle equal to $\frac{\pi}{2} - \theta$, we have by the first part,

$$dx'' = dx.\,\varphi\left(\frac{\pi}{2} - \theta\right) = \frac{y\,dx}{r}.$$

Therefore, equating these values, we obtain

$$d\theta = -\frac{y\,dx}{k r^2}.$$

In the same manner, by supposing y to increase and x to remain constant, we shall obtain the corresponding variation of the angle θ, by changing in the preceding equation x into y, and y into x, and θ into $\frac{\pi}{2} - \theta$, i. e. changing the sign of $d\theta$. Hence

$$d\theta = \frac{x\,dy}{k r^2}.$$

By means of these partial differentials, we find the total variation of the angle θ,

$$d\theta = \frac{x\,dy - y\,dx}{k r^2},$$

$$= \frac{1}{k}\cdot\frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Whence, by integration,

$$k\theta + \text{C} = \tan.^{-1}\frac{y}{x}.$$

$$\therefore \frac{y}{x} = \tan.\,(k\theta + \text{C}).$$

Combining this equation with the equation $r^2 = x^2 + y^2$, we find

$$x = r.\cos.\,(k\theta + \text{C}).$$

In order to determine the constants, let $y = 0$,

$$\therefore r = x, \text{ and } \theta = 0$$

$$\therefore \cos.\,\text{C} = 1, \text{ and } \text{C} = 0.$$

Again, let $x = 0$,

$$\therefore\ r = y, \text{ and } \theta = \frac{\pi}{2},$$

$$\therefore\ \cos.\, k\theta = 0,$$

and there k must equal $2n + 1$, n being any whole number, since x will $= 0$ as often as $\theta = \dfrac{\frac{\pi}{2}}{2n+1}$. But here $\theta = \dfrac{\pi}{2}$,

$$\therefore\ 2n + 1 = 1 \qquad \therefore\ n = 0.$$

Whence $x = r.\cos.\theta$.

The diagonal of the rectangle therefore not only represents the MAGNITUDE *but also the* DIRECTION *of the resultant.*

We can hence obtain the resultant of forces inclined at any angle to each other, as in art. 13.

II.

The proof given by Pontécoulant is, in the first part, very similar.

Since x and y are given in intensity and direction, we have

$$r = \mathrm{F}(x,y), \quad \theta = f(x,y);$$

whence, by eliminating y,

$$x = \varphi(r,\theta).$$

But since x and r are the only quantities dependent on the unit of force, their ratio $\frac{x}{r}$ ought to be independent of that unit, and therefore expressed as a simple function of θ, hence $x = r.\varphi\theta$, as before.

In order to determine the form of $\varphi\theta$, suppose a new force z to be introduced acting on the same point, and perpendicular to the plane of x and y. The resultant of these three forces may be obtained by first compounding any two, and then again compounding the resultant of these two with the third; and it is evident that the resultant will be the same in whatever order the composition be effected.

Let then r be the resultant of x and y,
θ its inclination to the direction of x.
Again, let r' be the resultant of r and z,
θ' its inclination to the direction of r.
Then, $x = r.\varphi\theta, \quad r = r'.\varphi\theta'.$

But, if we suppose r' to result from the composition of the resultant of y and z with x, and call ψ the angle which r' makes with the direction of x, we have

$$x = r'.\phi\psi.$$

By comparing this with the other equations, we find

$$\varphi\psi = \varphi\theta.\phi\theta'.$$

In order to deduce the value of $\varphi\theta$ from this equation, we must observe that θ and θ' are independent of each other, and therefore we may suppose them to vary separately. Hence, differentiating on this supposition, and putting

$$\frac{d.\varphi\theta}{d\theta} = \phi'\theta, \quad \frac{d.\varphi\theta'}{d\theta} = \varphi'\theta',$$

we have,

$$\frac{\frac{d\psi}{d\theta}}{\frac{d\psi}{d\theta'}} = \frac{\phi'\theta.\phi\theta'}{\phi\theta.\phi'\theta'},$$

an equation in which $\varphi\psi$ has disappeared.

Again, in the spherical triangle intercepted by the directions of the three forces x, r, r', the sides of which are θ, θ', ψ, we have

$$\cos.\,\psi = \cos.\,\theta.\,\cos.\,\theta'.$$

By differentiation, we obtain

$$\frac{d\psi}{d\theta} = \frac{\sin.\,\theta\,\cos.\,\theta'}{\sin.\,\psi}, \qquad \frac{d\psi}{d\theta'} = \frac{\cos.\,\theta\,\sin.\,\theta'}{\sin.\,\psi}.$$

Substituting these values of $\frac{d\psi}{d\theta}$, $\frac{d\psi}{d\theta}$, in the equation found above, we deduce,

$$\frac{\cos.\,\theta.\ \phi'\theta}{\sin.\,\theta.\ \phi\theta} = \frac{\cos.\,\theta'.\ \phi'\theta'}{\sin.\,\theta'.\ \phi\theta'}.$$

But since the two angles θ and θ' are independent, either member of this equation is constant, whatever value be assigned to the other variable, and therefore,

$$\frac{\cos.\,\theta.\,\varphi'\theta}{\sin.\,\theta.\,\varphi\theta} = k,$$

k being a constant independent of θ.

By integration we obtain,

$$\varphi\theta = \mathrm{C}\,(\cos.\,\theta)^{-k},$$

C being an arbitrary constant.

Substituting this value in the equation, $x = r.\varphi\theta$, we have

$$x = r\,\mathrm{C}\,(\cos.\,\theta)^{-k}.$$

To determine the constants C and k, let $y = 0$;

$$\therefore\ r = x, \qquad \text{and } \theta = 0;$$

and therefore

$$\cos.\,\theta = 0, \qquad \therefore\ \mathrm{C} = 1.$$

Again, suppose $x = y \therefore r = \sqrt{(x^2 + y^2)} = x\sqrt{2}$, and $\theta = \frac{\pi}{4}$.

$$\therefore x = x\sqrt{2}\left(\cos.\frac{\pi}{4}\right)^{-k}.$$

But, $\cos.\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \therefore k = -1$.

Hence $x = r\cos.\theta$.

Whence, &c. as before.

III.

Poisson's demonstration consists of the following propositions.

PROP. 1. *To find the resultant of two* EQUAL *forces acting on a point in directions inclined at any angle.*

Since the forces are equal there is no reason why the resultant should be nearer one rather than the other, and it will therefore bisect the angle contained by their directions.

We have therefore now to find its magnitude.

Let A B, A C (fig. 3, App.) represent the directions of the two forces which are each represented by p; let the angle B A C $= 2\theta$; and let A R, which bisects the angle, represent the direction of the resultant, R. The intensity of this resultant must depend on the values of p and θ, and therefore,

$$R = f(p, \theta).$$

But since R and p are the only quantities whose numerical value varies with the assumed unit of force, their ratio is independent of that unity, and must therefore be a simple function of θ, hence we obtain,

$$R = p . \varphi \theta,$$

and we have to determine the form of $\varphi \theta$.

For this purpose, draw through A the lines A B′, A B″, A C′, A C″, so that the angles, B′A B, B″A B, C′A C, C″A C are all equal to one another. Then the force represented by A B may be resolved into two equal forces acting in the directions A B′, A B″, and supposing p' the unknown value of each of these forces and ψ the angle B′A B, we have, as before,

$$p = p' . \varphi \psi,$$

ϕ denoting the same function as in the other equation.

The force acting in A C may be similarly resolved, and thus, instead of the two forces p we have four forces p', whose resultant ought to coincide, both in magnitude and direction, with the resultant of the two.

Now, if R′ be the resultant of the forces acting in the directions AB′, AC′, its direction must coincide with A R; and since the angle B′A R $=$ C′A R $= \theta - \psi$,

$$R' = p' . \varphi (\theta - \psi).$$

Similarly if R″ be the resultant of the forces acting in A B″, A C″,

$$R'' = p' . \varphi (\theta + \psi).$$

But these two *partial* resultants must make up the whole, therefore

$$R = R' + R''.$$

But, $$R = p . \phi \theta = p' . \phi \theta . \phi \psi,$$

$$\therefore \phi \theta . \phi \psi = \varphi (\theta - \psi) + \varphi (\theta + \psi).$$

Expanding $\phi(\theta - \psi)$ and $\phi(\theta + \psi)$ by Taylor's theorem, and dividing by $\phi\,\theta$, we obtain,

$$\phi\psi = 2\left\{1 + \frac{d^2.\phi\theta}{\phi\theta.d\theta^2}.\ \frac{\psi^2}{1.2} + \frac{d^4.\varphi\theta}{\varphi\theta.d\theta^4}.\ \frac{\psi^4}{1.2.3.4} + \&c.\right\}$$

But, since $\varphi\psi$ does not involve θ, that quantity cannot enter any of the coefficients,

$$\frac{d^2.\varphi\theta}{\varphi\theta.d\theta^2},\qquad \frac{d^4.\varphi\theta}{\varphi\theta.d\theta^4},\qquad \&c.$$

and they must therefore be constant, i. e. independent of the variables, θ and ψ. Let, then the first $= c$, and we have

$$\frac{d^2\phi\theta}{d\theta^2} = c.\phi\theta.$$

Whence, by differentiation, we obtain,

$$\frac{d^4.\varphi\theta}{d\theta^4} = c.\frac{d^2.\varphi\theta}{d\theta^2} = c^2.\varphi\theta,$$

$$\frac{d^6.\varphi\theta}{d\theta^6} = c^2.\frac{d^2.\varphi\theta}{d\theta^2} = c^3.\varphi\theta;$$

and consequently,

$$\varphi\psi = 2\left\{1 + \frac{c\psi^2}{1.2} + \frac{c^2.\psi^4}{1.2.3.4} + \frac{c^3\psi^6}{1.2.3.4.5.6} + \&c.\right\};$$

or if $c = - a^2$,

$$\varphi\psi = 2\left\{1 - \frac{a^2\psi^2}{1.2} + \frac{a^4\psi^4}{1.2.3.4} - \frac{a^6\psi^6}{1.2.3.4.5.6} + \&c.\right\};$$

This series is the development for the $\cos.\,a\,\psi$, and hence

$$\varphi\psi = 2\cos.\,a\,\psi.$$

This determines the form of the function denoted by φ, and hence,

$$R = p.\varphi\,\theta = 2\,p\cos.\,a\,\theta.$$

In order to determine the value of a, which is independent of θ, let $\theta = \frac{\pi}{2}$, then the two forces are directly opposed, and R $= 0$ $\therefore$ $\cos.\,a\,\frac{\pi}{2} = 0$, and a must be some odd whole number. Also $a = 1$; for if $a = 3$, then $R = 0$ when $\theta = \frac{\pi}{6}$, or when $2\,\theta = \frac{\pi}{3}$, or when the forces are not directly opposed. Similarly a cannot be any other odd number, and therefore

$$R = 2\,p\cos.\,\theta.$$

Hence *the resultant of two* equal *forces is represented in magnitude and direction by the diagonal of the parallelogram whose sides represent the two forces.*

Prop. 2. *To find the resultant of two* unequal *forces, acting on a point in directions* at right angles *to each other.*

Let A B, A C (fig. 4, App.), at right angles to each other, represent the forces: complete the parallelogram and draw the diagonals A D, B C cutting each other in O; draw through A the line E F parallel to B C, through B and C the lines B E and C F parallel to A D.

Then since A B C D is a rectangle, the diagonals A D, B C are equal; and $\therefore$ A E = A O = A F.

Hence the force A B may be considered as the resultant of the two equal forces A E, A O, and also A C as the resultant of A F, A O; therefore, instead of the two original forces, we have the four forces A E, A F, A O, A O. But the first two of these are equal and opposite, and consequently destroy each other; the other two conspire and form a resultant equal to 2 A O = A D.

Therefore the resultant of A B and A C is A D.

Prop. 3. *To find the resultant of* any *two forces acting on a point, in directions inclined at* any *angle to each other.*

Let A B, A C (fig. 5, App.) represent the forces; complete the parallelogram, and through A, B, and C, draw the lines E F, B H, and C K perpendicular, and also draw B E, C F parallel to the diagonal A D. Then since B H and C K are the altitudes of the equal triangles on the same base, B H = C K; and it easily appears that A H = K D.

Hence, by the last case, the force A B may be considered as the resultant of the two forces A E, A H at right angles, and also A C as the resultant of A F, A K: therefore, instead of the two original forces, we have the four A E, A F, A H, A K. But the first two are equal and opposite, and consequently destroy each other; the other two conspire and form a resultant equal to A H + A K = A D.

Therefore the resultant of A B and A C is A D.

Hence *the resultant of* any *two forces acting on a point is represented, both in* magnitude *and* direction, *by the diagonal of the parallelogram whose sides represent the forces.*

FINIS.

TALBOYS AND BROWNE, PRINTERS, OXFORD.

ERRATA.

Page 12, line 18, for A B, $\frac{p'}{p' - p''}$ read A B. $\frac{p'}{p' - p''}$

Page 122, line 4, for similar read similarly.

Page 152, last line, for $p + p$ read $p + p'$.

Fig. 1.

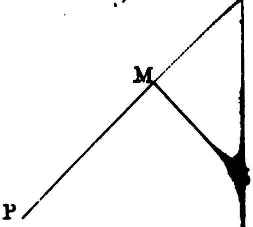


Fig. 4.

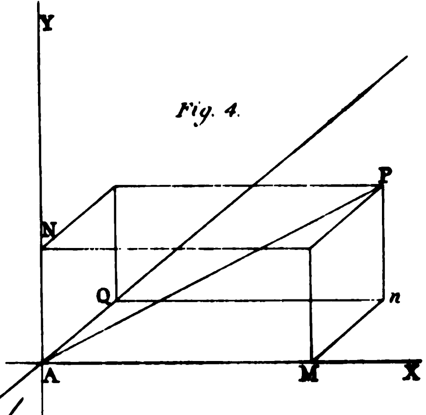


Fig. 2.

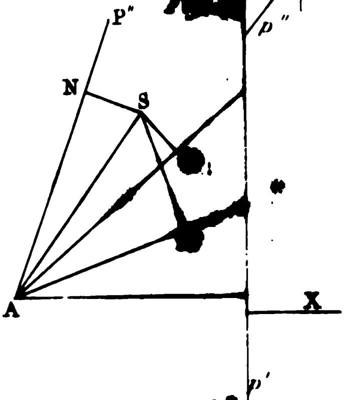


Fig. 9.

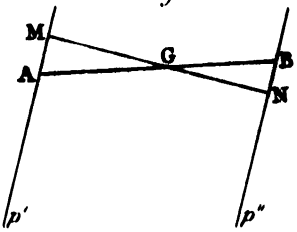


Fig. 6.

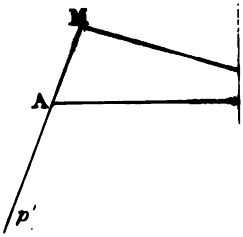


Fig. 12.

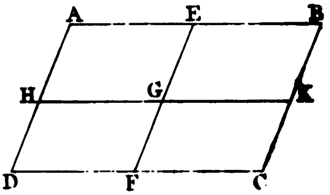


Fig. 13.

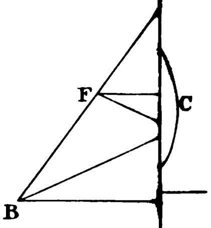


Fig. 17.

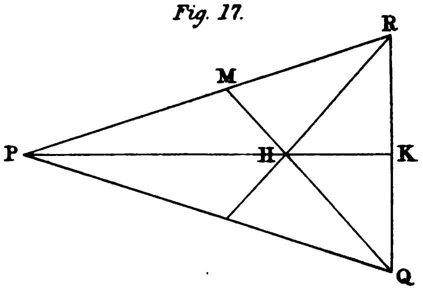


Fig. 18.

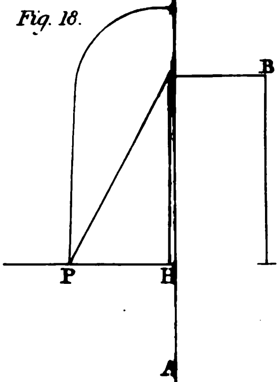


Fig. 21.

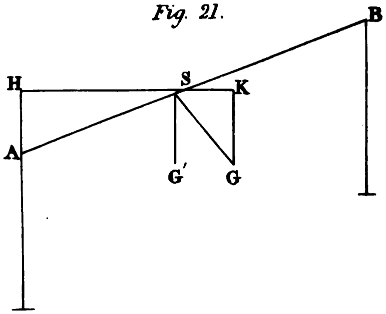


Fig. 24.

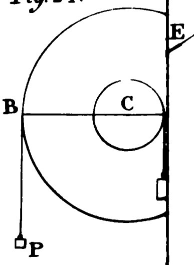


Fig. 26.

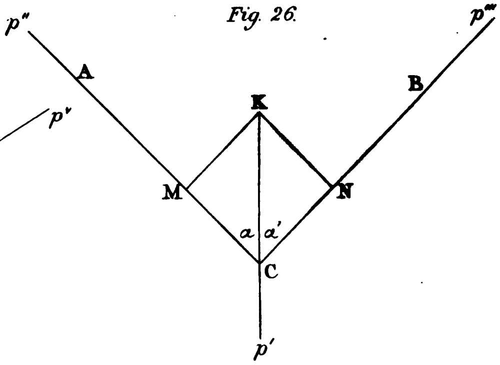


Fig. 13.

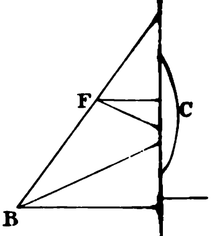


Fig. 17.

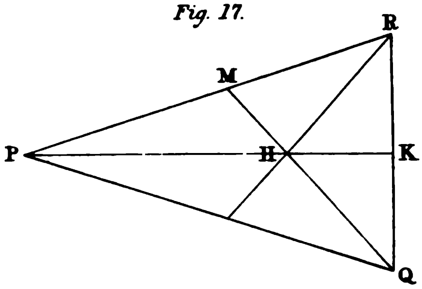


Fig. 18.

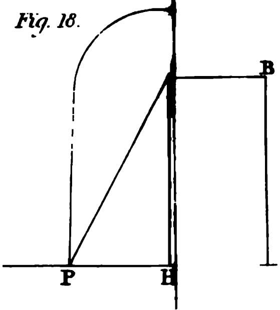


Fig. 21.

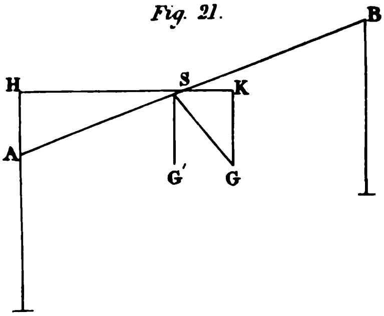


Fig. 24.

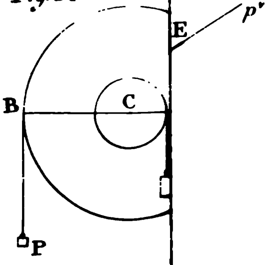


Fig. 26.

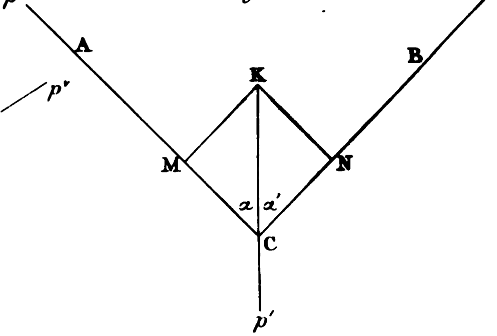


Fig. 30.

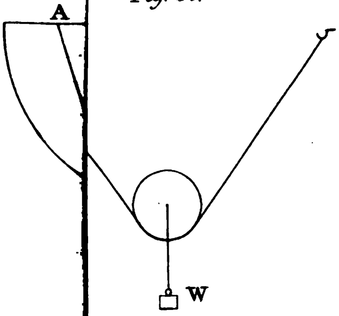


Fig. 31.

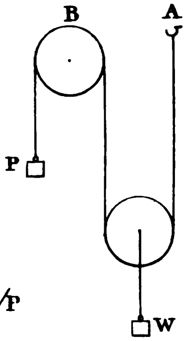


Fig. 32.

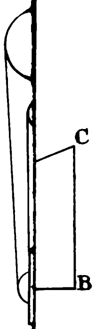
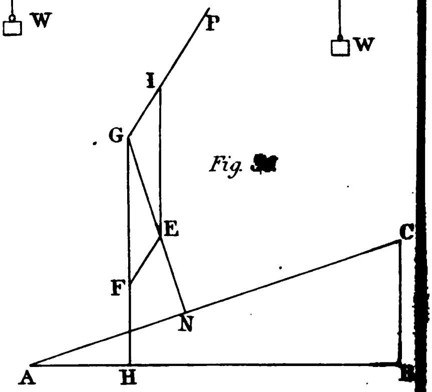


Fig. [illegible]



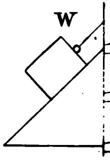
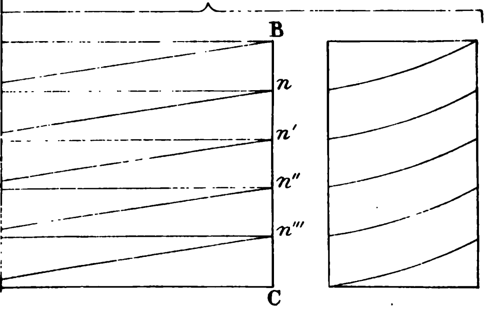


Fig. 40.



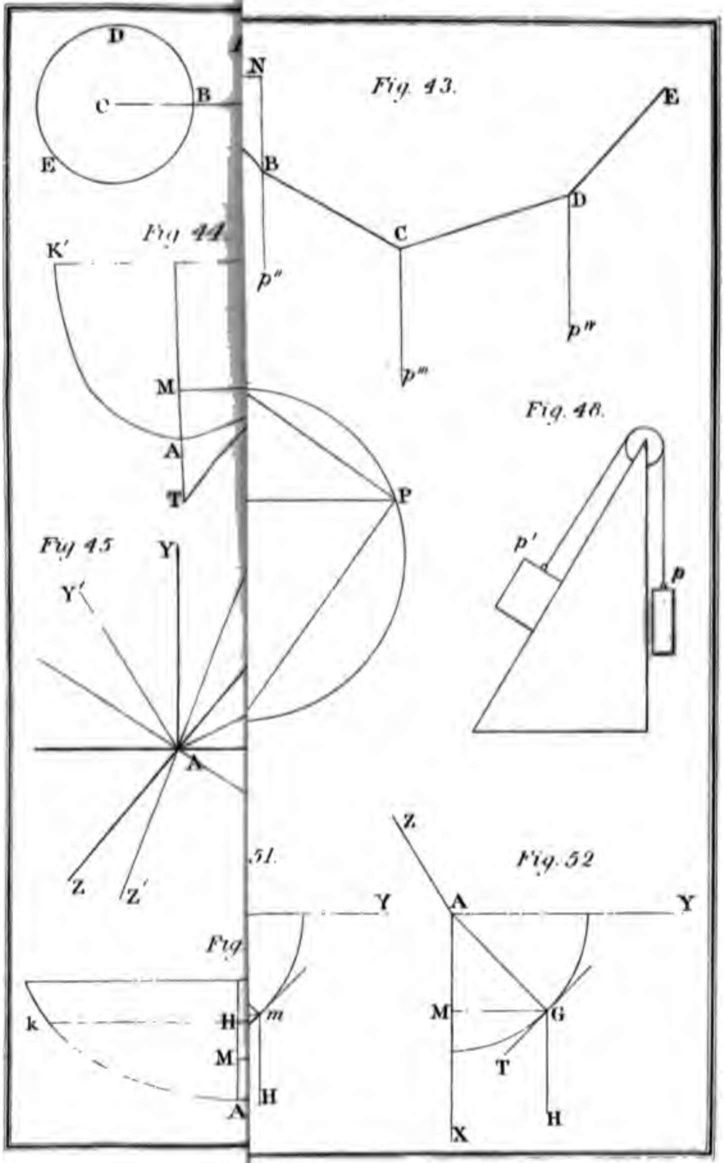
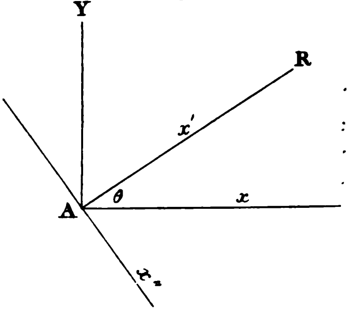

D
C
B
E
N
Fig. 43.
E
B
D
C
p″
p‴
p⁗
Fig. 44.
K′
M
A
T
P
Fig. 48.
p′
p
Fig. 45
Y
Y′
A
Z
Z′
51.
Fig.
Y
k
H
m
M
H
A
Z
Fig. 52
A
Y
M
G
T
H
X

Fig. 1.



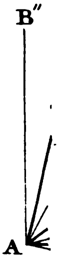


Fig. 4.